SECTION F

COMPOSITES

SECTION F1

COMPOSITES CONCEPTS

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

DEFINITION OF SYMBOLS

Symbol	Definition
$[A]$	extensional rigidity matrix of laminate - equation (F1. 2-18)
$[\bar{A}]$	average laminate extensional rigidity matrix - equation (F1. 2-28)
$[A^*]$	laminate compliance matrix - equation (F1. 2-30)
$[B]$	coupling matrix - equations (F1. 2-18) and (F1. 2-19)
$[C]$	lamina stiffness matrix - equation (F1. 1-6)
$[C^*]$	column form of the stiffness matrix - equation (F1. 1-27)
$[\bar{C}^*]$	column form of the transformed stiffness matrix - equation (F1. 1-28)
$[\bar{C}]$	transformed stiffness matrix of [C] - equation (F1. 2-15)
$[C']$	modified form of lamina stiffness matrix - equation (F1. 2-1)
$[\bar{C}']$	transformed matrix of the modified stiffness matrix - equation (F1. 2-10)
C_{ij}	components of the stiffness matrix $[C]$, $[C']$
$\bar{C}_{ij}$	components of the transformed stiffness matrix $[\bar{C}]$, $[\bar{C}*]$, $[\bar{C}']$
$[D]$	laminate flexural rigidity matrix - equation (F1. 2-19)
E	Young's modulus of elasticity
E_{ij}	Young's modulus of laminae in lamina principal direction - equation (F1. 1-15)

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
G_{ij}	shear moduli in the i-j plane - equation (F1. 1-15)
J	invariants of stiffness matrix transformations - equation (F1. 2-12)
K	lamina index indicating position in laminate
M	laminate bending and twisting moments - equation (F1. 2-17)
N	laminate forces - equation (F1. 2-16)
[S]	lamina compliance matrix - equation (F1. 1-7)
S_{ij}	components of the lamina compliance matrix
[T]	stress and strain transformation matrix - equation (F1. 1-3)
dA	differential area - equation (F1. 1-1)
h_K	distance from reference surface to plane separating laminae K and K+1 - Fig. F1. 2-3
t	laminate total thickness - Fig. F1. 2-3
x	principal longitudinal laminate axis (Fig. F1. 0-1)
y	principal transverse laminate axis (Fig. F1. 0-1)
z	principal normal laminate axis (Fig. F1. 0-1)
α	principal longitudinal lamina axis (Fig. F1. 0-2)
β	principal transverse lamina axis (Fig. F1. 0-2)
γ	shear strain

Symbol	Definition
γ^o	shear strain of laminate middle surface - equation (F1. 2-13)
ϵ	strain
ϵ^o	strain of laminate middle surface - equation (F1. 2-13)
$[\epsilon]$	strain column matrix - (F1. 1-10)
η	shear coupling ratios - equation (F1. 1-15)
θ	angular orientation of a lamina in a laminate, i. e., the angle between the x and α axis - positive is counterclockwise
ν	Poisson's ratio
σ	stress
$[\sigma]$	stress column matrix - equation (F1. 1-9)
τ	shear stress
χ	curvature of laminate middle surface - equation (F1. 2-13)

SUBSCRIPTS

1, 2, 3	arbitrary local coordinate system
α, β, z	lamina principal axes
x, y, z	laminate reference axes
ij	the (i-j) position in a sequence where i and j vary between 1 and 3

F1.0 COMPOSITES - BASIC CONCEPTS AND NOTATIONS.

The purpose of this section is to present state-of-the-art techniques utilized in the design and stress analysis of advanced composite structures. An attempt was made to keep the analytical developments and material as elementary as possible. However, the stress analysis of composite materials is more complex than that of conventional materials, and, as a result, the analysis techniques and concepts may seem rather involved.

In order to understand the mechanics of laminated composites, one must have a knowledge of certain basic definitions. These definitions, obtained primarily from References 1, 2, and 3, are intended to serve not only as a reference for this section but also as a guide to general literature on composite materials.

AEOLOTROPY	See anisotropic.
ANGLEPLY	Any filamentary laminate constructed with equal numbers of pairs of laminae with symmetry about the coordinate (x, y) axis. An alternate definition used frequently in current literature, but not in this section, is a laminate consisting of an even number of layers having the same thickness, and

the orthotropic axes of symmetry in each ply are alternately oriented at angles of + θ and - θ to the laminate axes.

ANISOTROPIC — Not isotropic; having mechanical and/or physical properties which vary with direction relative to natural reference axes inherent in the material.

BALANCED COMPOSITE — A composite laminate whose layup is symmetrical with relation to the midplane of the laminate (Fig. F1.0-1).

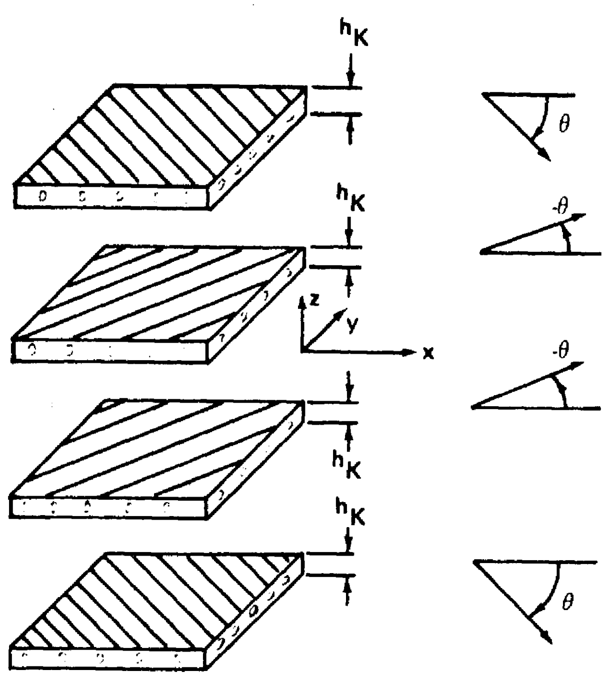

FIGURE F1.0-1. BALANCED OR SYMMETRIC COMPOSITE

BUCKLING

Buckling is a mode of failure characterized generally by an unstable lateral deflection caused by compressive action on the structural element involved. In advanced composites, buckling may take the form not only of conventional general instability and local instability but also of a microinstability of individual fibers.

COMPLIANCE MATRIX

The compliance matrix is defined by the equation $\epsilon_i = S_{ij}\sigma_j$, where S_{ij} are the components of the compliance matrix; may be obtained by inverting the stiffness matrix.

COMPOSITE MATERIAL

Composites are considered to be combinations of materials differing in composition or form on a macroscale. The constituents retain their identities in the composite; that is, they do not dissolve or otherwise merge completely into each other although they act in concert. Normally, the components can be physically identified and lead to an interface between components (Fig. F1.0-2).

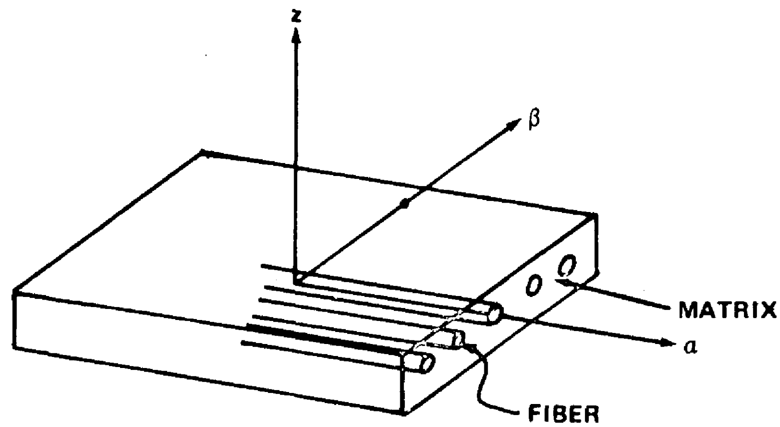

FIGURE F1.0-2. PRINCIPAL CONSTITUENTS OF COMPOSITE LAMINA AND PRINCIPAL AXES

CONSTITUENT — In general, an element of a larger grouping; in advanced composites, the principal constituents are the fibers and the matrix (refer to Fig. F1.0-2).

CONSTITUTIVE — Refers to the stress-strain (Hooke's Law) relationships for a material because the stress-strain relations actually describe the mechanical constitution of the material.

CROSSPLY — Any filamentary laminate constructed with equal numbers of pairs of laminae at angles of 0 deg and 90 deg to the laminate axes.

An alternate definition used frequently in current literature, but not in this section, is a laminate consisting of an even number of layers all of the same thickness with the orthotropic axes of symmetry in each ply alternately oriented at angles of 0 deg and 90 deg to the laminate axes.

DELAMINATION — The separation of the layers of material in a laminate.

FIBER — A single homogeneous strand of material, essentially one-dimensional in the macrobehavior sense, used as a principal constituent in advanced composites because of its high axial strength and modulus (refer to Fig. F1.0-2).

FIBER CONTENT — The amount of fiber present as reinforcement in a composite. This is usually expressed as a percentage volume fraction or weight fraction of the composite.

FIBER DIRECTION The orientation or alignment of the longitudinal axis of the fiber with respect to a stated reference axis (Fig. F1.0-3).

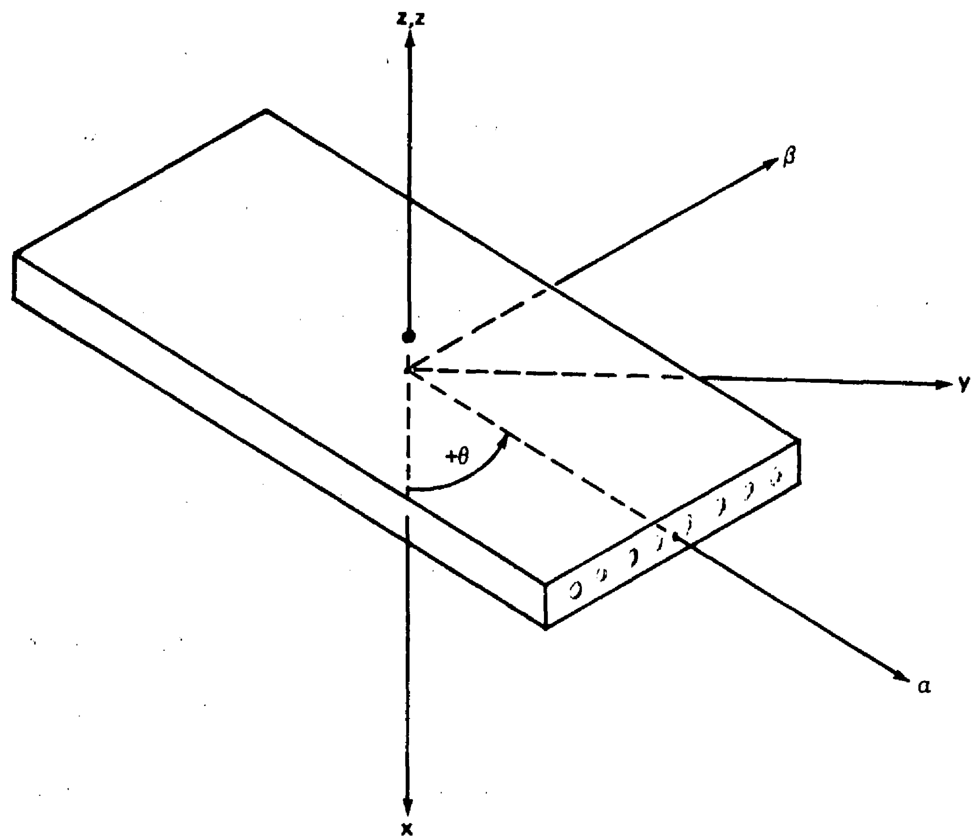

FIGURE F1.0-3. LAMINA AXIS ORIENTATION

FILAMENT A variety of fibers characterized by extreme length, such that there are normally no filament ends within a part except at geometric discontinuities. Filaments are used in filamentary composites and are also used in filament winding processes, which require long, continuous strands.

FILAMENTARY COMPOSITES

Composite materials of laminae in which the continuous filaments are in nonwoven, parallel, uniaxial arrays. Individual uniaxial laminae are combined into specifically oriented multiaxial laminates for application to specific envelopes of strength and stiffness requirements (Fig. F1.0-4).

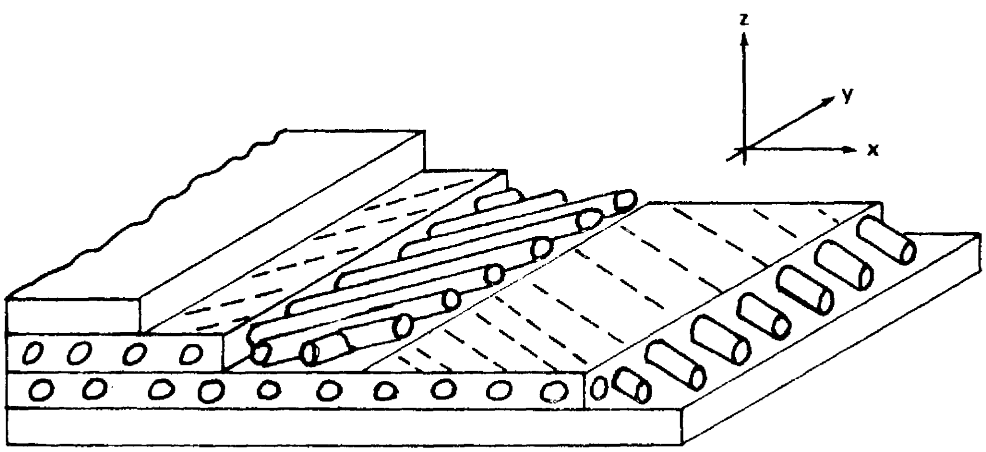

FIGURE F1.0-4. FILAMENTARY COMPOSITE

GENERALLY ORTHOTROPIC

Descriptive term for a lamina for which the constitutive equation, when transformed to an arbitrary set of axes, is fully populated. That is, for

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^K = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} \end{bmatrix}^K \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}^K ,$$

$\overline{C}_{16} \neq 0, \quad \overline{C}_{26} \neq 0,$

HENCKY-VON MISES DISTORTIONAL ENERGY THEORY

Yield criterion using distortional energy for isotropic materials:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 2\sigma_0^2 \quad .$$

HETEROGENEOUS

Descriptive term for a material consisting of dissimilar contituents separately identifiable; a medium consisting of regions of unlike properties separated by internal boundaries; not homogeneous.

HILL

Generalized von Mises yield criterion to account for anisotropy:

$$2f(\sigma_{ij}) = A_1(\sigma_2 - \sigma_3)^2 + A_2(\sigma_3 - \sigma_1)^2 + A_3(\sigma_1 - \sigma_2)^2 + 2A_4\sigma_{23}^2 + 2A_5\sigma_{31}^2 + 2A_6\sigma_{12}^2 = 1 ,$$

where

$2f(\sigma_{ij})$ = the plastic potential,

$$2A_1 = (F_2)^{-2} + (F_3)^{-2} - (F_1)^{-2} ,$$

$$2A_2 = (F_3)^{-2} + (F_1)^{-2} - (F_2)^{-2} ,$$

$$2A_3 = (F_1)^{-2} + (F_2)^{-2} - (F_3)^{-2} ,$$

$$2A_4 = (F_{23})^{-2} ,$$

$$2A_5 = (F_{31})^{-2} ,$$

$$2A_6 = (F_{12})^{-2} ,$$

and F_1, F_2, and F_3 are determined from uniaxial tension or compression tests, and F_{12}, F_{23}, and F_{31} are determined from pure shear tests.

HOMOGENEOUS — Descriptive term for a material of uniform composition throughout; a medium which has no internal physical boundaries; a material whose properties are constant and isotropic at every point.

HOMOGENEOUS ANISOTROPIC — Descriptive term for a material which has no plane of material symmetry such as the orthotropic material.

HOMOGENEOUS GENERALLY ORTHOTROPIC — Descriptive term for a lamina which behaves in a manner similar to the anisotropic lamina.

HOMOGENEOUS ISOTROPIC — Descriptive term for a lamina which has a constant modulus of elasticity and the $C_{16} = C_{26} = 0$ in its constitutive equation.

HORIZONTAL SHEAR — See interlaminar shear.

INTERFACE — The boundary between the individual, physically distinguishable constituents of a composite.

INTERLAMINAR SHEAR — Shear force which tends to produce a relative displacement between two laminae in a laminate along the plane of their interface.

ISOTROPIC — Descriptive term for a material which has uniform material properties in all directions.

LAMINA — A single ply or layer in a laminate made of a series of layers (Fig. F1.0-5).

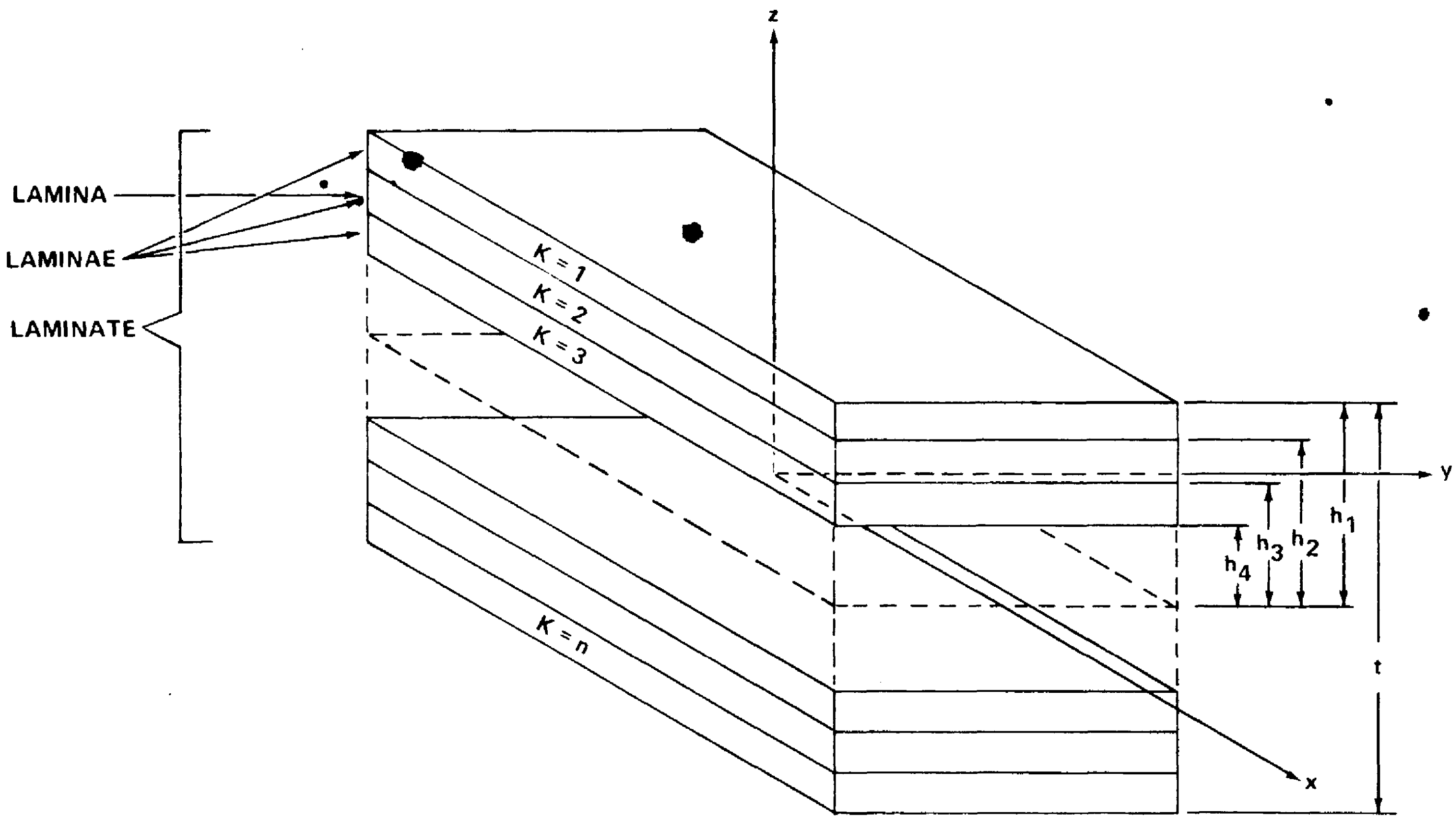

FIGURE F1.0-5. LAMINA NOTATION

LAMINAE	Plural of lamina (refer to Fig. F1.0-5).
LAMINATE	A product made by bonding together two or more layers of laminae of material or materials (refer to Fig. F1.0-5).
LAMINATE ORIENTATION	The configuration of crossplied composite laminate with regard to the angles of crossplying, the number of laminate at each angle, and the exact sequence of the individual laminae.
LAYUP	A process of fabrication involving the placement of successive layers of materials.
MACRO	In relation to composites, denotes the gross properties of a composite as a structural element but does not consider the individual properties or identity of the constituents.
MATRIX	The essentially homogeneous material in which the fibers or filaments of a composite are imbedded (refer to Fig. F1.0-2).

MICRO	In relation to composites, denotes the properties of the constituents and their effect on the composite properties.
ORTHOTROPIC	Descriptive term for a material which has three mutually perpendicular planes of elastic symmetry.
PRINCIPAL AXES	The set of axes in a lamina which is parallel and perpendicular to the filament direction is called the lamina principal axes (refer to Fig. F1.0-2).
QUASI-ISOTROPIC	Descriptive term for a laminate which has essentially isotropic stiffnesses and perhaps strength.
SPECIALLY ORTHOTROPIC	Descriptive term for lamina for which the $C_{16} = C_{26} = 0$ in its constitutive equation.
TENSOR	A tensor is a physical entity in nature which obeys certain transformation relations. There are different orders of

tensors, and each order has its own transformation relations.

TRANSVERSELY ISOTROPIC — Descriptive term for a material exhibiting a special case of orthotropy in which properties are identical in two orthotropic dimensions, but not in the third; having identical properties in both transverse directions but not the longitudinal direction.

VON-MISES DISTORTIONAL ENERGY THEORY — See Hencky-von Mises Distortional Energy Theory.

x-AXIS — An axis in the plane of the laminate which is used as the 0 deg reference for designating the angle of lamina (refer to Fig. F1.0-5).

y-AXIS — The axis in the plane of the laminate which is perpendicular to the x-axis (refer to Fig. F1.0-5).

z-AXIS — The reference axis normal to the plane of the laminate (refer to Fig. F1.0-5).

1.1 BASIC CONCEPTS.

Some of the basic concepts applicable to all continuums, in particular composites, are presented in this subsection. These include the concepts of stress and strain at a point and their transformation relations.

1.1.1 STRESS AND STRAIN.

Following the guide of Reference 1, stress and strain relations at a point will be reviewed to form a firm base for the analytical development for composites.

A tensor, as defined previously, is some physical entity in nature which obeys certain transformation relations. A scalar, for example, is a tensor of zero-th order, and a vector is a tensor of first order. It is well known that the components of a vector change when the coordinate system is altered or rotated. This change in the components of the vector is governed by certain mathematical relations or transformations. Each order of tensors has its own transformation relations; therefore, it is necessary only to establish that a physical entity is a tensor and determine its order, and the transformation relations are defined.

Stress and strain are both second-order tensors and their transformation relations are well known (the graphical form of the transformation is the Mohr's circle). These transformation relations may be derived from the equilibrium relations of a small element. Consider a two-dimensional problem as shown in Fig. F1.1-1. By summing forces in the "1" direction, the following equation results:

$$\sigma_1 \, dA - \sigma_x \, (\cos\theta \, dA) \, (\cos\theta) - \sigma_y \, (\sin\theta \, dA) \, (\sin\theta)$$

$$- \tau_{xy} \, (\sin\theta \, dA) \, (\cos\theta) - \tau_{xy} \, (\cos\theta \, dA) \, (\sin\theta) = 0 \, ,$$

or

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \, (2 \sin\theta \cos\theta) \qquad \text{(F1.1-1)}$$

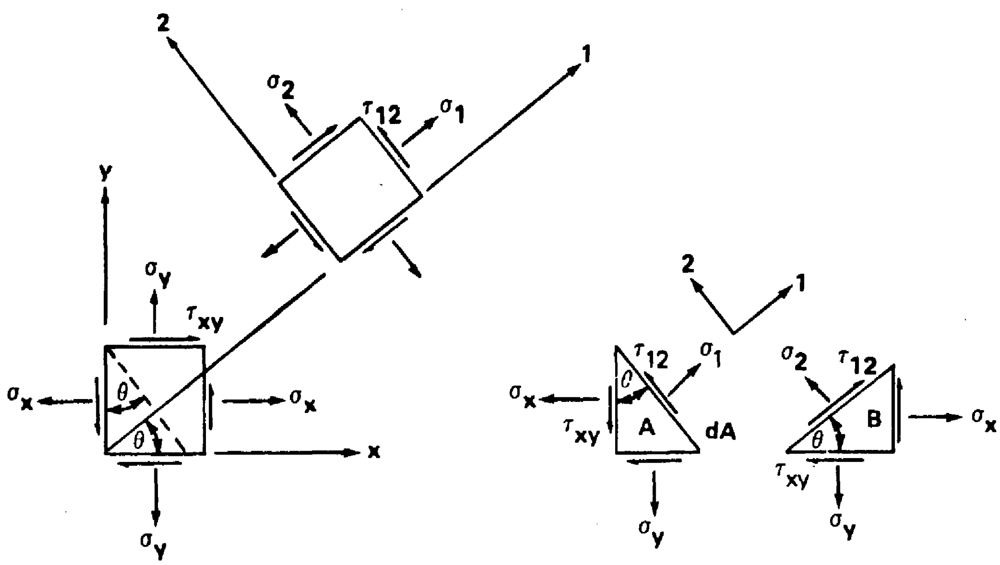

FIGURE F1.1-1. STRESS COORDINATE ROTATION

In a similar manner, the other transformed stresses, σ_2 and τ_{12}, may be determined. These equations may be written in a form convenient for later developments, a matrix form. Thus,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & (2\sin\theta\cos\theta) \\ \sin^2\theta & \cos^2\theta & (-2\sin\theta\cos\theta) \\ (-\sin\theta\cos\theta) & (\sin\theta\cos\theta) & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} . \quad \text{(F1.1-2)}$$

Using a more compact notation, we may write equation (F1.1-2) as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \text{(F1.1-3)}$$

where [T] is the symbol for the transformation matrix. Equation (F1.1-3) is the transformation relation for the stress tensor when reduced to a two-dimensional space. Equation (F1.1-3) is the necessary relationship required to transform any two-dimensional stress state from one set of coordinates to another set.

With a slight modification, the two-dimensional strain may be transformed by the same transformation:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = [T] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix} . \quad \text{(F1.1-4)}$$

1.1.2 GENERALIZED HOOKE'S LAW.

In this paragraph the constants of proportionality between stress and strain (Hooke's law constants) are shown to be components of a fourth-order tensor and therefore have a set of transformation relations different from those for stress and strain. Several forms of the Hooke's law relationships and the elastic constants will be shown for the various material conditions.

1.1.2.1 Homogeneous Isotropic Material.

For the familiar homogeneous isotropic material in a one-dimensional stress state [1], the Hooke's law relationship is

$$\sigma = E\epsilon \ . \qquad \text{(F1.1-5)}$$

The proportionality constant (E) is Young's modulus, or the modulus of elasticity, and is a scalar value.

1.1.2.2 Elastic Linear Anisotropic Material.

Consider the most general material, but require elasticity and linearity, which is the anisotropic material. This material has 21 elastic constants. The constitutive equation (Hooke's law) is [2]

$$[\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Symmetric} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} [\epsilon] \ . \qquad \text{(F1.1-6)}$$

The components C_{ij} are called components of the "stiffness" matrix. The equation may be written as

$$[\epsilon] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ \text{Symmetric} & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} [\sigma] \tag{F1.1-7}$$

where

$$[C] = [S]^{-1} . \tag{F1.1-8}$$

The components S_{ij} are called components of the "compliance" matrix. The $[\sigma]$ conventionally symbolizes [3]

$$[\sigma] = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} . \tag{F1.1-9}$$

Similarly,

$$[\epsilon] = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} . \tag{F1.1-10}$$

1.1.2.3 Monoclinic Material.

If a material possesses one plane of symmetry, it is termed a monoclinic material and has 13 independent elastic constants. If the plane of symmetry is assumed to be the x-y plane, the constitutive equations are [2]

$$[\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} [\epsilon] \qquad \text{(F1.1-11)}$$

and

$$[\epsilon] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & & & S_{44} & S_{45} & 0 \\ & \text{Symmetric} & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} [\sigma]. \qquad \text{(F1.1-12)}$$

1.1.2.4 Orthotropic Material.

If the anisotropic material possesses two orthogonal planes of symmetry, assuming $x = 0$ and $z = 0$, the material is termed orthotropic. In this condition, there are only nine independent elastic constants. Note that if a material has two orthogonal planes of symmetry, three orthogonal planes of symmetry exist. The constitutive equations of the orthotropic material are

$$[\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ \text{Symmetric} & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} [\epsilon] \tag{F1.1-13}$$

and

$$[\epsilon] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ \text{Symmetric} & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} [\sigma] \ . \tag{F1.1-14}$$

Since most composite structures will be constructed of orthotropic laminates, this material is of particular interest. The engineering constants, values which are establishable from uniaxial and pure shear tests, may easily be equated to the components of the compliance matrix. The compliance components are then

$$S_{11} = \frac{1}{E_{11}} , \qquad S_{22} = \frac{1}{E_{22}} , \qquad S_{33} = \frac{1}{E_{33}} .$$

$$S_{12} = \frac{-\nu_{12}}{E_{11}} \qquad S_{23} = \frac{-\nu_{23}}{E_{22}} \qquad S_{31} = \frac{-\nu_{31}}{E_{33}}$$

$$= \frac{-\nu_{21}}{E_{22}} , \qquad = \frac{-\nu_{32}}{E_{33}} , \qquad = \frac{-\nu_{13}}{E_{11}} . \tag{F1.1-15}$$

$$S_{66} = \frac{1}{G_{12}} , \qquad S_{55} = \frac{1}{G_{13}} \qquad S_{44} = \frac{1}{G_{23}} .$$

$$S_{16} = \frac{\eta_{16}}{E_{11}} , \qquad S_{26} = \frac{\eta_{26}}{E_{22}} , \qquad S_{36} = \frac{\eta_{36}}{E_{33}} .$$

where

E_{11}, E_{22}, E_{33} = Young's moduli in the 1, 2, and 3 (x, y, and z) directions, respectively,

ν_{ij} = Poisson's ration

$= \dfrac{\text{strain in the j direction}}{\text{strain in the i direction}}$

caused by a stress in the i direction,

G_{ij} = shear moduli in the i-j plane,

η_{ij} = shear coupling ratios.

Note that the S_{16}, S_{26}, and S_{36} terms are used for a "monoclinic" material. From equation (F1.1-8), the components of the stiffness matrix may be determined as

$$
\begin{aligned}
C_{11} &= (1 - \nu_{23}\,\nu_{32})\ VE_{11} \ , \\
C_{22} &= (1 - \nu_{31}\,\nu_{13})\ VE_{22} \ , \\
C_{33} &= (1 - \nu_{12}\,\nu_{21})\ VE_{33} \ , \\
C_{12} &= (\nu_{21} + \nu_{23}\,\nu_{31})\ VE_{11} \ , \\
&= (\nu_{12} + \nu_{13}\,\nu_{32})\ VE_{22} \ , \\
C_{13} &= (\nu_{31} + \nu_{21}\,\nu_{32})\ VE_{11} \ , \\
&= (\nu_{13} + \nu_{23}\,\nu_{12})\ VE_{33} \ , \\
C_{23} &= (\nu_{32} + \nu_{12}\,\nu_{31})\ VE_{22} \ , \\
&= (\nu_{23} + \nu_{21}\,\nu_{13})\ VE_{33} \ , \\
C_{44} &= G_{23} \ , \\
C_{55} &= G_{31} \ ,
\end{aligned}
\qquad \text{(F1.1-16)}
$$

and

$$C_{66} = G_{12} \ ,$$

where

$$V = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31})^{-1} \quad .$$

For an orthotropic material in a state of plane stress, the constitutive equation is [1]

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \qquad \text{(F1. 1-17)}$$

where

$$C_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \quad ,$$

$$C_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \quad , \qquad \text{(F1. 1-18)}$$

$$C_{12} = \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})} = \frac{\nu_{12}E_{22}}{(1 - \nu_{12}\nu_{21})} \quad ,$$

and

$$C_{66} = G_{12} \quad .$$

1.1.2.5 Isotropic Material.

For an isotropic material, there are only two independent elastic constants. The constitutive relations are [3]

$$[\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ & \text{Symmetric} & & & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} [\epsilon] \qquad (F1.1\text{-}19)$$

and

$$[\epsilon] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ \text{Symmetric} & & S_{11} & 0 & 0 & 0 \\ & & & 2(S_{11} - S_{12}) & 0 & 0 \\ & & & & 2(S_{11} - S_{12}) & 0 \\ & & & & & 2(S_{11} - S_{12}) \end{bmatrix} [\sigma] \; . \qquad (F1.1\text{-}20)$$

The constitutive constants may be defined in terms of the engineering constants as

$$C_{11} = C_{22} = C_{33} = \frac{(1 - \nu)\, E}{(1 + \nu)\,(1 - 2\nu)} \;,$$

$$C_{12} = C_{13} = C_{23} = \frac{\nu E}{(1 + \nu)\,(1 - 2\nu)} \;, \qquad (F1.1\text{-}21)$$

$$C_{44} = C_{55} = C_{66} = G = \frac{E}{2\,(1 + \nu)} \;,$$

$$S_{11} = S_{22} = S_{33} = \frac{1}{E} \;,$$

$$S_{12} = S_{13} = S_{23} = \frac{-\nu}{E} \;,$$

and

$$S_{44} = S_{55} = S_{66} = \frac{1}{G} = \frac{2(1 + \nu)}{E} \quad .$$

If the isotropic material is assumed to be in a two-dimensional stress state (plane stress), equation (F1.1-19) may be written as [1]

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \qquad \text{(F1.1-22)}$$

where

$$C_{11} = C_{22} = \frac{E}{(1 - \nu^2)} \quad ,$$

$$C_{12} = \frac{E}{(1 - \nu^2)} \quad , \qquad \text{(F1.1-23)}$$

and

$$C_{66} = \frac{E}{2(1 + \nu)} = G \quad .$$

1.1.2.6 Transformation of Stiffness Matrix.

In Reference 1, the elastic constants (stiffness) for a material are stated to be components of a fourth-order tensor and consequently, must obey certain transformation relations. The transformation of a general anistropic material, in three dimensions and rotated an angle θ about the z-axis, is given in Reference 2 as

$$[\overline{C}*] = [T_R] \; [C*] \tag{F1.1-24}$$

where

$$[T_R] = \left[\begin{array}{cccccc|cccccc|ccc|ccc|cc|c}
m^4 & 2m^2n^2 & 2m^3n & n^4 & 2mn^3 & m^2n^2 & & & & & & & & & & & & & & & \\
m^2n^2 & m^4+n^4 & mn^3-m^3n & m^2n^2 & m^3n-mn^3 & -m^2n^2 & & & & & & & & & & & & & & & \\
-2m^3n & 2m^3n-2mn^3 & m^4-3m^2n^2 & 2mn^3 & 3m^2n^2-n^4 & m^3n-mn^3 & & & & & & & & & & & & & & & \\
n^4 & 2m^2n^2 & -2mn^3 & m^4 & -2m^3n & m^2n^2 & & & & & & & & & & & & & & & \\
-2mn^3 & 2mn^3-2m^3n & 3m^2n^2-n^4 & 2m^3n & m^4-3m^2n^2 & mn^3-m^3n & & & & & & & & & & & & & & & \\
4m^2n^2 & -8m^2n^2 & 4mn^3-4m^3n & 4m^2n^2 & 4m^3n-4mn^3 & (m^2-n^2)^2 & & & & & & & & & & & & & & & \\
\hline
 & & & & & & m^3 & -m^2n & mn^2 & -n^3 & m^2n & -mn^2 & & & & & & & & & \\
 & & & & & & m^2n & m^3 & n^3 & mn^2 & mn^2 & m^2n & & & & & & & & & \\
 & & & & & & mn^2 & -n^3 & m^3 & -m^2n & -m^2n & mn^2 & & & & & & & & & \\
 & & & & & & n^3 & mn^2 & m^2n & m^3 & -mn^2 & -m^2n & & & & & & & & & \\
 & & & & & & -2m^2n & 2mn^2 & 2m^2n & -2mn^2 & m^3-mn^2 & n^3-m^2n & & & & & & & & & \\
 & & & & & & -2mn^2 & -2m^2n & 2mn^2 & 2m^2n & m^2n-n^3 & m^3-mn^2 & & & & & & & & & \\
\hline
 & & & & & & & & & & & & m^2 & n^2 & mn & & & & & & \\
 & & & & & & & & & & & & n^2 & m^2 & -mn & & & & & & \\
 & & & & & & & & & & & & -2mn & 2mn & m^2-n^2 & & & & & & \\
\hline
 & & & & & & & & & & & & & & & m^2 & -2mn & n^2 & & & \\
 & & & & & & & & & & & & & & & mn & m^2-n^2 & -mn & & & \\
 & & & & & & & & & & & & & & & n^2 & 2mn & n^3 & & & \\
\hline
 & & & & & & & & & & & & & & & & & & m & -n & \\
 & & & & & & & & & & & & & & & & & & n & m & \\
\hline
 & & & & & & & & & & & & & & & & & & & & 1
\end{array}\right] \tag{F1.1-25}$$

The m and n terms represent

$$m = \cos\theta$$

and (F1.1-26)

$$n = \sin\theta \; .$$

The [C*] and [$\overline{C}$*] are column matrix forms of equation (F1.1-6) and the transformed equation. The [C*] is defined as

$$[C^*] = \begin{bmatrix} C_{11} \\ C_{12} \\ 2C_{16} \\ C_{22} \\ 2C_{26} \\ 4C_{66} \\ C_{14} \\ C_{15} \\ C_{24} \\ C_{25} \\ 2C_{46} \\ 2C_{56} \\ C_{13} \\ C_{23} \\ 2C_{36} \\ C_{44} \\ C_{45} \\ C_{55} \\ C_{34} \\ C_{35} \\ C_{33} \end{bmatrix} \qquad (F1.1\text{-}27)$$

and the $[\overline{C}*]$ is defined as

$$[\overline{C}*] = \begin{bmatrix} \overline{C}_{11} \\ \overline{C}_{12} \\ 2\overline{C}_{16} \\ \overline{C}_{22} \\ 2\overline{C}_{26} \\ 4\overline{C}_{66} \\ \overline{C}_{14} \\ \overline{C}_{15} \\ \overline{C}_{24} \\ \overline{C}_{25} \\ 2\overline{C}_{46} \\ 2\overline{C}_{56} \\ \overline{C}_{13} \\ \overline{C}_{23} \\ 2\overline{C}_{36} \\ \overline{C}_{44} \\ \overline{C}_{45} \\ \overline{C}_{55} \\ \overline{C}_{34} \\ \overline{C}_{35} \\ \overline{C}_{33} \end{bmatrix} \qquad \text{(F1.1-28)}$$

The transformation of the stiffness matrix for an orthotropic material in a plane stress state is of particular interest since this will be of direct applicability to fibrous composites. As shown in Reference 1, when the elastic constants are needed with respect to some axis other than the material axis, the transformed elastic constants are

$$
\begin{aligned}
\overline{C}_{11} &= C_{11}\cos^4\theta + 2\,[C_{12} + 2C_{66}]\sin^2\theta\cos^2\theta + C_{22}\sin^4\theta\ , \\
\overline{C}_{22} &= C_{11}\sin^4\theta + 2\,[C_{12} + 2C_{66}]\sin^2\theta\cos^2\theta + C_{22}\cos^4\theta\ , \qquad \text{(F1.1-29)} \\
\overline{C}_{12} &= [C_{11} + C_{22} - 4C_{66}]\sin^2\theta\cos^2\theta + C_{12}\,[\sin^4\theta + \cos^4\theta]\ , \\
\overline{C}_{66} &= [C_{11} + C_{22} - 2C_{12} - 2C_{66}]\sin^2\theta\cos^2\theta + C_{66}\,[\sin^4\theta + \cos^4\theta]\ , \\
\overline{C}_{16} &= [C_{11} - C_{12} - 2C_{66}]\sin\theta\cos^3\theta + [C_{12} - C_{22} + 2C_{66}]\sin^3\theta\cos\theta\ ,
\end{aligned}
$$

and

$$
\overline{C}_{26} = [C_{11} - C_{12} - 2C_{66}]\sin^3\theta\cos\theta + [C_{12} - C_{22} + 2C_{66}]\sin\theta\cos^3\theta\ .
$$

The angle of rotation (θ) is about the z-axis and assumed positive in the counterclockwise direction.