and

$$\frac{8}{9}(1+\nu)\ A_{11} - \pi^2 ab\ B_{22}\left(\frac{1}{b^2} + \frac{1-\nu}{2a^2}\right) = 0 \quad .$$

For any plate dimensions and material properties, the stresses may then be determined by using the appropriate design equations. This gives

$$\sigma_x = \frac{E}{1-\nu^2}\left(A_{11}\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \nu\frac{2\pi}{b}B_{22}\sin\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right)$$

$$- \frac{E\alpha}{1-\nu}T_0\left(1-\frac{x}{a}\right) \quad ,$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\frac{2\pi}{b}B_{22}\sin\frac{2\pi x}{a}\cos\frac{2\pi y}{b} + \nu\frac{\pi}{a}A_{11}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right)$$

$$- \frac{E\alpha}{(1-\nu)}T_0\left(1-\frac{x}{a}\right) \quad ,$$

and

$$\tau_{xy} = G'\left(A_{11}\frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} + B_{22}\frac{2\pi}{a}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{a}\right) \quad .$$

It can be seen that, for more complicated temperature distributions and higher values of N, efficiency considerations would dictate the use of a relatively simple digital computer program in applying this method of analysis.

3.0.8 Shells.

The analysis of shells subjected to temperature variations has, for the most part, taken the approach of treating thermal loadings as equivalent mechanical loadings and hence solving the stresses and displacements by techniques such as in Section B7.3. These approaches are discussed in Refs. 7 and 8. Some of the more common temperature distributions in shells will be discussed in the following section.

3.0.8.1 Isotropic Circular Cylindrical Shells.

This section covers the thermostructural analysis of thin-walled, right circular, isotropic cylindrical shells. The middle-surface curvilinear coordinate axes (x and y) are always taken parallel to the axis of revolution and the circumferential direction, respectively.

The organization of this section is somewhat different from that of the sections which cover isotropic flat plates. This is due to certain fundamental differences between the physical behavior of flat plates and shells. Flat-plate deformations are of such a nature that it is helpful to group the solutions for stable constructions into the following categories:

1. Temperature gradients through the thickness

2. Uniform temperatures through the thickness.

Except for the special case of self-equilibrating gradients through the thickness $(N_T = 0,\ M_T = 0)$, the first of these two cases involves out-of-plane bending which is, of course, accompanied by displacements normal to the middle surface of the undeformed plate. In case 1, the plate remains flat; that is, the only displacements occur in directions parallel to the original middle surface and no out-of-plane bending occurs. The governing differential equations in these two instances are quite different and the indicated separation of the cases is a logical format for the sections dealing with flat plates. However, the situation is not the same for circular cylindrical shell structures. For these components, a single governing differential equation includes the phenomena related to both cases 1 and 2 and there is no need to isolate these two types of thermal conditions. This is because either type of temperature distribution, in conjunction with clamped or simply supported boundaries, will lead to both membrane loading and bending about the shell-wall middle surface. Consequently, for stable constructions which comply with either case 1 or 2, the solutions are given in a single grouping as follows.

I. Analogies with Isothermal Problems.

A. Configuration.

This discussion is restricted to thin-walled, right circular cylinders which are of constant thickness and are made of isotropic material. It is assumed that the shell wall is free of holes and that it obeys Hooke's law.

Figure 3.0-37 depicts the isotropic cylindrical shell configuration. Figure 3.0-38 shows the sign convention for forces, moments, and pressure.

B. Boundary Conditions.

The following three types of boundary conditions are discussed:

1. Clamped edge; that is,

$$w = \frac{dw}{dx} = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \tag{1}$$

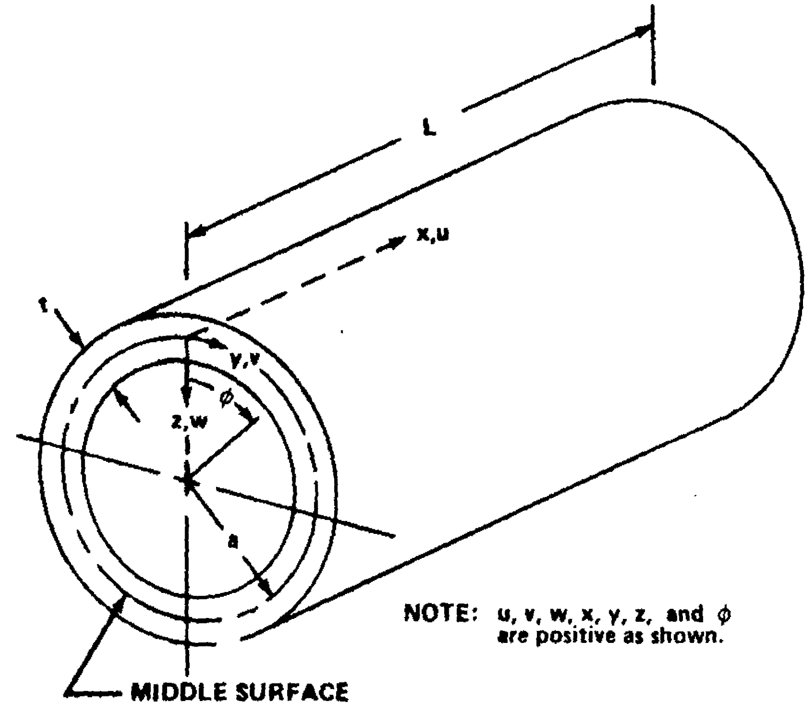


Figure 3.0-37. Isotropic cylindrical shell configuration for analogies with isothermal problems.

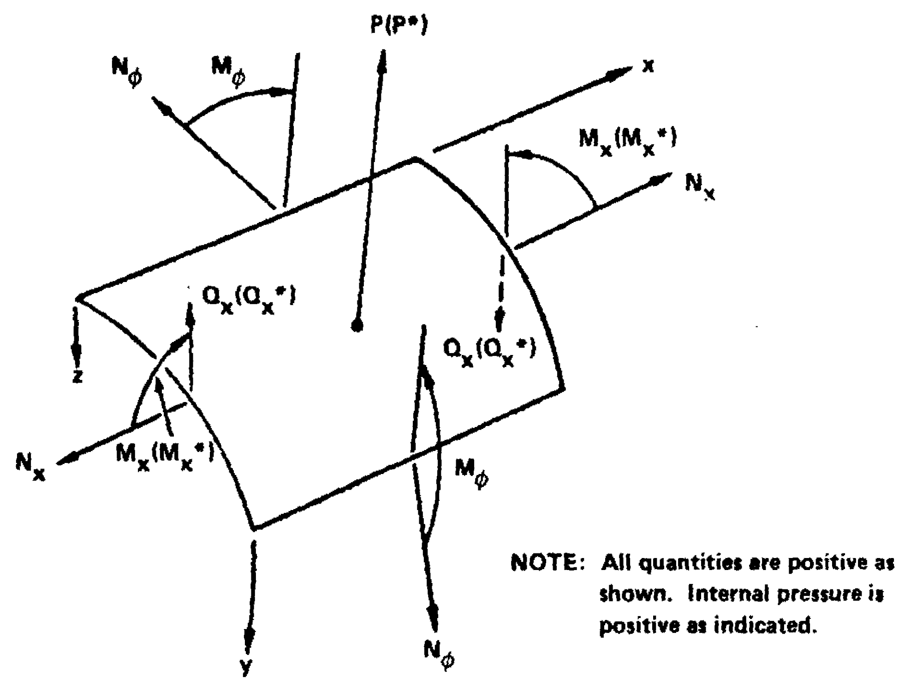


Figure 3.0-38. Sign convention for forces, moments, and pressure for analogies with isothermal problems.

2. Simply supported edge; that is,

$$w = M_x = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \tag{2}$$

3. Free edge; that is,

$$M_x = Q_x = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{3}$$

All possible combinations of these boundary conditions are permitted. Hence, it is not required that those at $x = 0$ be the same as those at $x = L$. In every case, it is assumed that the cylinder is unrestrained in the axial direction ($N_x = 0$).

C. Temperature Distribution.

The temperature distribution must be axisymmetric but arbitrary gradients may be present both through the wall thickness and in the axial direction. The permissible distributions can therefore be expressed in the form

$$T = T(x, z) \quad . \tag{4}$$

Any of the special cases for this equation are acceptable, including that where the entire shell is at constant temperature.

D. Analogies.

It is helpful for the user to recognize that, for circular cylinders, analogies exist between problems involving axisymmetric temperature distributions and certain problems where mechanical loading is present but thermal effects are entirely absent (isothermal problems). The various types of correspondence are discussed herein where it is assumed that Young's modulus and Poisson's ratio are unaffected by temperature changes. On the other hand, one can account for temperature dependence of the thermal-expansion coefficient α by observing that it is the product αT which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α. When this approach is taken, the user must recognize that any reference to linear temperature distribution is actually a reference to a straight-line variation of the product αT.

It is also helpful for the user to recognize that, regardless of the complexity of a thermal gradient through the thickness, at any location (x, y) the distribution can be resolved into

1. A self-equilibrating component, and/or
2. A uniform component, and/or
3. A nonuniform linear component passing through $T = 0$ at the middle surface.

A self-equilibrating temperature component is one for which

$$\int_{-t/2}^{t/2} T \, dz = 0$$

and

$$\int_{-t/2}^{t/2} Tz \, dz = 0 \quad . \qquad (5)$$

An example of such a distribution is illustrated in Figure 3.0-39.

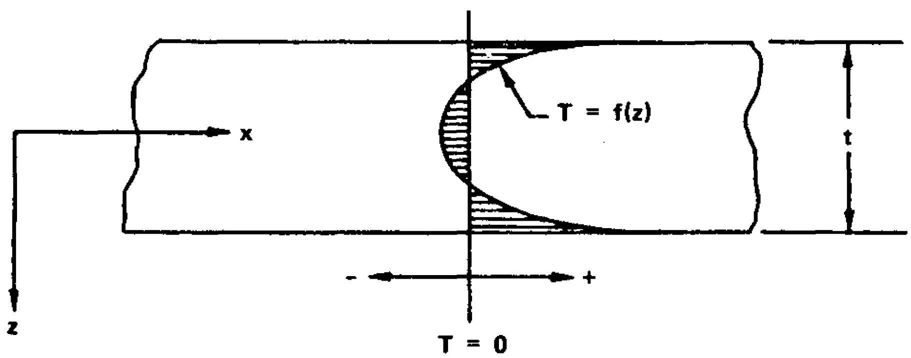


Figure 3.0-39. Sample self-equilibrating temperature distribution.

From a practical viewpoint, it may be assumed that gradients of this type will not cause any deformation of the cylinder. Their only influence will be on the stresses σ_x and σ_ϕ. If, for example, a solution is available for any arbitrary temperature distribution $T = T(x, z)$, the effects can be easily superimposed from a component $T_1(x, z)$ which satisfies equations (5). It is necessary only to algebraically add the stresses

$$\sigma_x = \sigma_\phi = -\frac{E\alpha T_1}{(1 - \nu)} \qquad (6)$$

to the previously determined values for the appropriate locations (x, z).

In this section, the following two categories are treated separately:

1. Uniform temperatures through the thickness with or without axisymmetric, longitudinal gradients; that is,

$$T = T(x) \quad . \tag{7}$$

2. Nonuniform linear temperature gradients through the thickness, passing through $T = 0$ at the middle surface with or without axisymmetric longitudinal variations and possibly including self-equilibrating components; that is,

$$T = T_1(x, z) + T_2(x) \frac{z}{t} \tag{8}$$

where T_1 satisfies equations (5).

<u>Uniform Temperatures Through the Thickness $[T = T(x)]$</u>.

In Ref. 16, Tsui presents the small-deflection governing differential equation for the subject shell. After conversion to the notation and sign convention used here, this expression becomes

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{p}{D_b} - \frac{N_T}{D_b a} - \frac{1}{D_b(1-\nu)} \frac{d^2 M_T}{dx^2} \tag{9}$$

where

$$\beta = \sqrt[4]{\frac{Et}{4D_b a^2}} = \sqrt[4]{\frac{3(1-\nu^2)}{a^2 t^2}} \quad ,$$

$$N_T = E\alpha \int_{-t/2}^{t/2} T\,dz \quad ,$$

$$M_T = E\alpha \int_{-t/2}^{t/2} Tz\,dz \quad , \tag{10}$$

and

$$D_b = \frac{Et^3}{12(1-\nu^2)} \quad .$$

An inspection of equation (9) reveals the key to an analogy which applies to the problem under discussion. Note that, for the isothermal problem, this equation reduces to

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{p}{D_b} \quad . \tag{11}$$

For thermostructural problems where pressure differentials are absent and the temperature is uniform through the wall thickness ($M_T = 0$), one obtains

$$\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{N_T}{D_b a} \quad . \tag{12}$$

A comparison of equations (11) and (12) suggests that the latter problem may be treated by means of an isothermal model that is loaded by a pressure differential, $p = p^*$, where

$$p^*(x) = \frac{N_T}{a} = \frac{E\alpha Tt}{a} \quad . \tag{13}$$

With regard to the edges, it should be noted that, since $M_T = 0$, the boundary conditions for the actual cylinder can be expressed as follows [16]:

1. Clamped edge:

$$w = \frac{dw}{dx} = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{14}$$

2. Simply supported edge:

$$w = \frac{d^2w}{dx^2} = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{15}$$

3. Free edge:

$$\frac{d^2w}{dx^2} = \frac{d^3w}{dx^3} = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{16}$$

These relationships do not contain any temperature terms and are therefore identical to those for the corresponding isothermal cases. Therefore, the major equivalence between the subject temperature distribution $[T = T(x)]$ and the isothermal model is bound up in the governing differential equation (9). Hence, the desired analogy is achieved by simply substituting p^* for p in equation (11). When this pressure is positive, it acts radially outward. It is important to note, however, that the analogy is complete only insofar as the radial deflections and the axial stresses are concerned; that is, the thermal deflections w and the thermal stresses σ_x will be identical to those due solely to a pressure p^* acting on a cylinder having the same geometry and boundary conditions as the actual structure. On the other hand, to determine the thermal stress σ_ϕ, the quantity $(-E\alpha T)$ must be added to the corresponding value obtained from the pressure solution. This accounts for the fact that strains ϵ_ϕ in the amount αT are associated with stress-free thermal growths or shrinkages.

To facilitate the application of this analogy, the user should refer to Section 2.40 of Ref. 17, which includes solutions for numerous cases of pressure-loaded cylindrical shells (a wide variety of pressure distributions and boundary conditions are treated).

Temperature Gradients Through the Thickness [$T = T(x,z)$].

This subsection discusses factors associated with cylindrical shells having nonuniform linear temperature gradients through the thickness, passing through $T = 0$ at the middle surface, with or without axisymmetric longitudinal variations, and possibly including self-equilibrating components; that is,

$$T = T_1(x, z) + T_2(x) \frac{z}{t} \tag{17}$$

where

$$\int_{-t/2}^{t/2} T_1 \, dz = 0$$

and

$$\int_{-t/2}^{t/2} T_1 z \, dz = 0 \quad . \tag{18}$$

Here again, it is helpful for the user to study the following small-deflection governing differential equation which was obtained by converting the corresponding formulation of Ref. 16 to the notation and sign convention of this section:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{p}{D_b}\frac{N_T}{D_b a} - \frac{1}{D_b(1-\nu)}\frac{d^2M_T}{dx^2} \tag{19}$$

where

$$\beta = \sqrt[4]{\frac{Et}{4D_b a^2}} = \sqrt[4]{\frac{3(1-\nu^2)}{a^2t^2}} \quad ,$$

$$N_T = E\alpha \int_{-t/2}^{t/2} T \, dz \quad , \tag{20}$$

$$M_T = E\alpha \int_{-t/2}^{t/2} Tz \, dz \quad ,$$

and

$$D_b = \frac{Et^3}{12(1-\nu^2)} \quad .$$

For the isothermal problem, this equation reduces to

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{p}{D_b} \tag{21}$$

while, for the subject thermostructural problem,

$$N_T = 0 \tag{22}$$

and equation (19) becomes

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{1}{D_b(1-\nu)} \frac{d^2M_T}{dx^2} \quad . \tag{23}$$

A comparison of equations (21) and (23) suggests that the latter problem may be treated by means of an isothermal model that is loaded by a pressure differential, $p = p^*$, where

$$p^*(x) = \frac{1}{(1-\nu)} \frac{d^2M_T}{dx^2} \quad . \tag{24}$$

This does not, however, provide a complete basis for the desired analogy since the boundary conditions which are likewise part of the problem formulation must be considered. From a study of conventional types of boundaries, it is clear that the simplest form of the subject analogy is that associated with a cylinder having both ends clamped. In this case, both the radial deflection and the related slope must vanish at the boundaries; that is,

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned}\right\} \quad \text{at } x = 0 \text{ and } x = L \quad . \tag{25}$$

The simplicity of the analogy for this situation comes about because these relationships do not contain any temperature terms and are therefore identical to those for the corresponding isothermal problem. As a result, the major equivalence between the subject temperature distribution and the pressure loading is bound up in the governing differential equation (19). Hence for cylinders clamped at both ends, the analogy is achieved by simply substituting p^* for p in equation (21). When this pressure is positive, it acts radially outward. It is important to note, however, that the analogy is complete only insofar as the radial deflections are concerned. That is, the thermal deflections w will be identical to those due solely to a pressure p^* acting on a cylinder having the same geometry and boundary conditions as the actual structure. On the other hand, to determine the thermal stresses σ_x and σ_ϕ, the quantity $[-E\alpha T/(1 - \nu)]$ must be added to each of the corresponding values obtained from the pressure solution. This accounts for the possible presence of a self-equilibrating temperature component and for the fact that strains ϵ_x and ϵ_ϕ, in the amount αT, are due dolely to stress-free thermal growths or shrinkages.

Although the analogy under discussion takes on its simplest form where both boundaries are clamped, this general method need not be ruled out for a simply supported shell. In the latter case, the analogy with respect to equation (19) still holds true. The added complexity is introduced only through the boundary-condition formulations. In this connection, the bending moment M_x may be expressed as follows [16]:

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1-\nu)} \quad . \tag{26}$$

This relationship is applicable anywhere in the shell, including positions around the boundaries. Only the latter locations need be considered here. It should be recalled that the condition of simple support includes the requirement that

$$M_x = 0 \qquad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{27}$$

Suppose a uniformly distributed external bending moment $M_x{}^*$ is applied around such a boundary, as defined by the equation

$$M_x{}^* = \frac{M_T}{(1-\nu)} \quad . \tag{28}$$

The user must remember that the sign convention being used specifies that a positive bending moment causes compressive stresses on the outer surface of the shell wall (refer to Fig. 3.0-38). By superimposing moments $M_x{}^*$ around a simply supported end, the related expressions for the boundary conditions become

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1-\nu)} + \frac{M_T}{(1-\nu)} = 0 \quad \text{at } x = 0 \text{ and/or } x = L \tag{29}$$

or

$$M_x = -D_b \frac{d^2w}{dx^2} = 0 \quad \text{at } x = 0 \text{ and/or } x = L \quad . \tag{30}$$

Equation (30) is the same as that for a simply supported boundary of a cylinder which does not experience any thermal influences. Hence, for a circular, cylindrical shell having both ends simply supported and subjected to the temperature distribution defined by equations (17) and (18), thermal deflections w can be obtained by superposition of the following:

1. The radial deflections of a simply supported cylinder which is identical to the actual structure but is free of any thermal influences and is subjected to a pressure p^* where

$$p^*(x) = \frac{1}{(1-\nu)} \frac{d^2M_T}{dx^2} \tag{31}$$

and bursting pressures are positive.

2. The radial deflections of a cylinder which is free of thermal influences, is identical to the actual structure, and whose boundaries conform with the condition of simple-support except for the application of uniformly distributed moments M_x^* at each end, where

$$M_x^* = \frac{M_T}{(1 - \nu)} \quad . \tag{32}$$

The thermal stresses σ_x and σ_ϕ are found by adding the quantity $[-E\alpha T/(1 - \nu)]$ to the algebraic sum of these stresses associated with steps 1 and 2. To facilitate the application of this analogy, the user should refer to Section 2.40 of Ref. 17, which includes numerous solutions that will often be useful in performing step 1. In addition, Refs. 17, 11, and 18 provide simple methods by which step 2 may be accomplished.

A situation similar to the foregoing arises when both boundaries of the shell are free. As before, the analogy with respect to equation (19) remains valid but still greater complexity is introduced through the applicable boundary-condition formulations. In this case, when the cylinder is subjected to the temperature distribution defined by equations (17) and (18), thermal deflections w can be obtained by superposition of the following:

1. The radial deflections of a cylinder which has both ends free and is identical to the actual structure but is free of any thermal influences and is subjected to a pressure p^* where

$$p^*(x) = \frac{1}{(1 - \nu)} \frac{d^2 M_T}{dx^2} \tag{33}$$

and bursting pressures are positive.

2. The radial deflection of a cylinder which is free of thermal influences, is identical to the actual structure, and has the prescribed boundary conditions except for the application of uniformly distributed moments $M_x{}^*$ at each end, where

$$M_x{}^* = \frac{M_T}{(1 - \nu)} \quad . \tag{34}$$

3. The radial deflection of a cylinder which is free of thermal influences, is identical to the actual structure, and has the prescribed boundary conditions except for the application of uniformly distributed shear forces $Q_x{}^*$ at each end, where

$$Q_x{}^* = \frac{1}{(1 - \nu)} \frac{dM_T}{dx} \quad . \tag{35}$$

The thermal stresses σ_x and σ_ϕ are found by adding the quantity $[-E\alpha T/(1 - \nu)]$ to the algebraic sum of these stresses associated with items 1 and 2. To facilitate the application of this analogy, the user should refer to Section 2.40 of Ref. 17, which includes numerous solutions that will often be useful in the accomplishment of step 1. In addition, Refs. 17, 11, and 18 provide simple methods by which steps 2 and 3 may be accomplished.

Analogies of the types presented in this section perform a two-fold function in that they can help the user to develop some physical insight into thermostructural behavior and also enable him to solve thermal stress problems by using existing solutions for members subjected solely to mechanical loading. Although the emphasis has been on cases where both ends of the cylinder have the same boundary conditions, the user should find it relatively easy to apply the same basic concepts when the two boundaries are not identical (for example, one end clamped while the other is simply supported).

II. Thermal Stresses and Deflections — Linear Radial Gradient, Axisymmetric Axial Gradient.

A. Configuration.

The design equations provided here apply only to long $(L \geq 2\pi/\beta)$, thin-walled, right circular cylinders which are of constant thickness and are made of isotropic material. It is assumed that the shell wall is free of holes and that it obeys Hooke's law. Figure 3.0-37 depicts the isotropic cylindrical shell configuration. Figure 3.0-38 shows the sign convention used for the stress resultants of interest.

B. Boundary Conditions.

The following types of boundary conditions are discussed:

1. Free edges
2. Simply supported edges
3. Clamped edges.

All possible combinations of these boundary conditions are permitted; that is, it is not required that those at $x = 0$ be the same as those at $x = L$. However, in every case, it is assumed that the cylinder is unrestrained in the axial direction ($N_x = 0$).

C. Temperature Distribution.

The following types of temperature distributions may be present:

1. A radial gradient which is linear through the wall thickness and need not vanish at the middle surface

2. Axisymmetric axial gradients.

The permissible distributions can therefore be expressed in the form

$$T = T_1(x) + T_2(x) \frac{z}{t} \quad . \tag{36}$$

Certain restrictions must sometimes be imposed on the complexities of the functions $T_1(x)$ and $T_2(x)$, depending upon the method of analysis employed. These conditions are explained in a subsequent paragraph. Any of the special cases for equation (36) are acceptable; that is, either or both of $T_1(x)$ and $T_2(x)$ can be finite constants and either can be equal to zero.

D. Design Equations.

Throughout this section, it is assumed that Young's modulus and Poisson's ratio are unaffected by temperature changes. Hence, the user must select single effective values for each of these properties by employing some type of averaging technique. The same approach may be taken with regard to the coefficient of thermal expansion. On the other hand, the temperature-dependence of this property can be accounted for by recognizing that it is the product of αT which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α. When this approach is taken, the user must recognize that any reference to a linear temperature distribution is actually a reference to a straight-line variation of the product αT.

In addition, the several types of solutions cited here are based on classical small-deflection theory. Therefore, it is important for the user to be aware of this when applying the given methods to pressurized cylinders by superimposing the thermal stresses and deflections upon the corresponding values due solely to pressure; that is, because of their dependence upon classical theory, the methods presented here cannot account for nonlinear coupling between thermal deflections and the pressure-related meridional loads.

The small-deflection governing differential equation for the subject cylindrical shell (refer to Fig. 3.0-37) can be written as follows [19]:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{N_T}{D_b a} - \frac{1}{D_b(1-\nu)} \frac{d^2 M_T}{dx^2} \tag{37}$$

where

$$\beta = \sqrt[4]{\frac{Et}{4D_b a^2}} = \sqrt[4]{\frac{3(1-\nu^2)}{a^2 t^2}} ,$$

$$N_T = E\alpha \int_{-t/2}^{t/2} T\,dz ,$$

$$M_T = E\alpha \int_{-t/2}^{t/2} Tz\,dz , \qquad (38)$$

and

$$D_b = \frac{Et^3}{12(1-\nu^2)} .$$

If $L \geq 2\pi/\beta$, then overlapping effects from boundary conditions at the two ends of the cylinder can be neglected and the following approximate solutions to equation (37) apply to each half of the structure:

<u>CASE I ($0 \leq x \leq L/2$)</u>:

$$w = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) - \frac{aN_T}{Et} - \frac{a^2}{Et(1-\nu)} \frac{d^2M_T}{dx^2} . \qquad (39)$$

<u>CASE II ($L/2 \leq x \leq L$)</u>:

$$w = e^{-\beta(L-x)}[C_3 \cos \beta(L-x) + C_4 \sin \beta(L-x)] - \frac{aN_T}{Et} - \frac{a^2}{Et(1-\nu)} \frac{d^2M_T}{dx^2} . \qquad (40)$$

The terms $e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x)$ and $e^{-\beta(L-x)}[C_3 \cos \beta(L - x) + C_4 \sin \beta(L - x)]$ comprise the respective complementary solutions to equation (37). All of the boundary-condition influences are embodied in these terms. The remaining portions of equations (39) and (40) are the so-called particular solutions and those given here are somewhat inexact. They were obtained by Przemieniecki [20] as first-order approximations from an asymptotic integration process and will give exact results only if the functions $T_1(x)$ and $T_2(x)$ are truly polynomials of second degree or lower; for example, $T_1(x) = b_0 + b_1x + b_2x^2$ and $T_2(x) = d_0 + d_1x + d_2x^2$.

When the temperature distribution is such that polynomial expansions of T_1 and/or T_2 require terms higher than the second degree, approximate solutions can be obtained by using either of the following two procedures:

1. Truncate the series by eliminating terms having exponents greater than two and perform the analysis as though the resulting series were exact representations of $T_1(x)$ and $T_2(x)$.

2. Ignore the stated restriction on the polynomial expansions and use the actual higher-degree formulations.

Either of these two possibilities will introduce inaccuracies and, at present, no studies have been performed to determine the orders of magnitudes for the errors associated with various ranges of the parameters involved. For those cases where accurate results are required but equations (39) and (40) are inexact, the user can always resort to an alternative procedure whereby these expressions are suitably modified. This can be accomplished by retaining the foregoing complementary solutions but introducing more appropriate particular solutions. The latter can be established by standard mathematical operations such as variation of parameters or the method of undetermined coefficients [21, 22].

For any case, the constants C_1 through C_4 in the deflection relationships must be evaluated from the boundary conditions. It therefore becomes necessary to express the various physical possibilities by means of the following formulas:

1. Free edge:

$$Q_x = M_x = 0 \quad .$$

2. Simply supported edge:

$$w = M_x = 0 \quad . \tag{41}$$

3. Clamped edge:

$$w = \frac{dw}{dx} = 0 \quad .$$

where

$$Q_x = \frac{dM_x}{dx}$$

and (42)

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1-\nu)} \quad .$$

The method to be used will be illustrated by using the example of a cylinder having a simply supported edge at $x = 0$ and a clamped edge at $x = L$. For the portion of the cylinder where $0 \le x \le L/2$, the values $x = 0$, $w = 0$ would be inserted into equation (39) to obtain a relationship which may be identified as equation (39a). Following this, equation (39) must be substituted into

$$D_b \frac{d^2w}{dx^2} + \frac{M_T}{(1-\nu)} = 0 \tag{43}$$

and x must then be set equal to zero in the resulting formulation to obtain an equation which may be identified as (39b). The constants of integration C_1 and C_2 can then be determined by the simultaneous solution of equations (39a) and (39b). For the portion of the cylinder where $L/2 \le x \le L$, the values $x = L$, $w = 0$ would be inserted into equation (40) to obtain a relationship which may be identified as (40a). Following this, equation (40) must be substituted into

$$\frac{dw}{dx} = 0 \tag{44}$$

and x must then be set equal to L in the resulting formulation to obtain an equation which may be identified as (40b). The constants of integration C_3 and C_4 can then be determined by the simultaneous solution of equations (40a) and (40b).

Once the deflection equations have been found for both halves of the cylinder, the bending moments M_x and M_ϕ at any point can be established from

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1 - \nu)}$$

and (45)

$$M_\phi = -\nu D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1 - \nu)} \quad .$$

Then the stresses at any location are given by the following:

$$\sigma_x = \frac{12z}{t^3} M_x$$

and (46)

$$\sigma_\phi = \frac{N_\phi}{t} + \frac{12z}{t^3} M_\phi$$

where

$$N_\phi = -N_T - \frac{w}{a} Et \quad . \tag{47}$$

Example No. 1.

A thin-walled, right circular cylinder of length $L(> 2\pi/\beta)$ has both ends simply supported and is subjected to the temperature distribution

$$T = d_0 \frac{z}{t} \tag{48}$$

where d_0 is a constant. It is desired that the deflections and extreme-fiber stresses be found, assuming that $L = 40$ in. and $\beta = 0.2$.

For the given temperature distribution, equations (38) yield

$$N_T = 0$$

and (49)

$$M_T = \frac{E\alpha t^2 d_0}{12} \quad .$$

Then, for $0 \leq x \leq L/2$, equation (39) becomes

$$w = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) \quad .$$

At $x = 0$, the boundary condition of simple support requires that

$$w = M_x = 0 \tag{51}$$

where

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1 - \nu)} \quad . \tag{52}$$

From equations (50), (51), and (52), the constants C_1 and C_2 are found to be

$$C_1 = 0$$

and (53)

$$C_2 = \frac{\alpha t a^2 \beta^2}{6(1 - \nu)} d_0 \quad .$$

Hence, equation (50) may now be rewritten as follows:

$$w = \frac{\alpha t a^2 \beta^2}{6(1 - \nu)} d_0 e^{-\beta x} \sin \beta x \quad . \tag{54}$$

The function $\zeta(\beta x)$, tabulated by Timoshenko and Woinowsky-Krieger [11] may now be introduced to obtain

$$w = \frac{\alpha t a^2 \beta^2}{6(1 - \nu)} d_0 \zeta(\beta x) \tag{55}$$

where

$$\zeta(\beta x) = e^{-\beta x} \sin \beta x \quad . \tag{56}$$

By substituting equation (55) into (45), the bending moments can be expressed as

$$M_x = \frac{E\alpha t^2 d_0}{12(1-\nu)} [\theta(\beta x) - 1]$$

and (57)

$$M_\phi = \frac{E\alpha t^2 d_0}{12(1-\nu)} [\nu\theta(\beta x) - 1]$$

where the function $\theta(\beta x)$ is tabulated by Timoshenko and Woinowsky-Krieger [11] and is defined as follows:

$$\theta(\beta x) = e^{-\beta x} \cos \beta x \quad . \tag{58}$$

The extreme-fiber stresses can then be determined from

$$\sigma_x = \mp \frac{6M_x}{t^2}$$

and (59)

$$\sigma_\phi = \frac{N_\phi}{t} \mp \frac{6M_\phi}{t^2}$$

where the upper signs correspond to the outermost fibers. By substituting equation (55) and the first of equations (49) into (47), the following expression is obtained for N_ϕ :

$$N_{\phi} = - \frac{E\alpha t^2 a\beta^2}{6(1 - \nu)} d_0 \zeta(\beta x) \quad . \tag{60}$$

For the other half of the cylinder ($L/2 \le x \le L$), equation (40) must be used, which, in view of equations (49), becomes

$$w = e^{-\beta(L-x)} [C_3 \cos \beta(L - x) + C_4 \sin \beta(L - x)] \quad . \tag{61}$$

At $x = L$, the boundary condition of simple support requires that

$$w = M_x = 0 \tag{62}$$

where

$$M_x = -D_b \frac{d^2w}{dx^2} - \frac{M_T}{(1 - \nu)} \quad . \tag{63}$$

From equations (61), (62), and (63), the constants C_3 and C_4 are found to be

$$C_3 = 0$$

and (64)

$$C_4 = \frac{\alpha t a^2 \beta^2}{6(1 - \nu)} d_0 \quad .$$

Then, proceeding in the same manner as for the other half of the cylinder, the following expressions are found for the deflections, moments, and extreme-fiber stresses:

$$w = \frac{\alpha t\, a^2\beta^2}{6(1-\nu)}\, d_0\, \zeta[\beta(L-x)] \quad , \tag{65}$$

$$M_x = \frac{E\alpha t^2 d_0}{12(1-\nu)}\, \{\theta[\beta(L-x)] - 1\} \quad ,$$

(66)

$$M_\phi = \frac{E\alpha t^2 d_0}{12(1-\nu)}\, \{\nu\theta[\beta(L-x)] - 1\} \quad ,$$

$$\sigma_x = \mp \frac{6M_x}{t^2} \quad ,$$

and (67)

$$\sigma_\phi = \frac{N_\phi}{t} \mp \frac{6M_\phi}{t^2}$$

where

$$N_\phi = -\frac{E\alpha t^2\, a\beta^2}{6(1-\nu)}\, d_0\, \zeta[\beta(L-x)] \quad . \tag{68}$$

Here again, the upper signs in the stress formulas correspond to the outermost fibers.

The foregoing relationships for the two halves of the cylinder were used to obtain the nondimensional solution listed in Table 3.0-7. These results are plotted in Figure 3.0-40.

Example No. 2.

A thin-walled, right circular cylinder of length $L(> 2\pi/\beta)$ has both ends clamped and is subjected to the temperature distribution

$$T = b_0 + b_1 x \tag{69}$$

TABLE 3.0-7. NONDIMENSIONAL TABULAR SOLUTION FOR w AND M_x (EXAMPLE NO. 1)[a,b,c]

$0 \le x \le L/2$					$L/2 \le x \le L$					
x	βx	$\zeta(\beta x)$	$\theta(\beta x)$	$\theta(\beta x)-1$	x	(L-x)	$\beta(L-x)$	$\zeta[\beta(L-x)]$	$\theta[\beta(L-x)]$	$\theta[\beta(L-x)]-1$
0	0	0	1.000	0	20.0	20.0	4.0	-0.0139	-0.0120	-1.0120
0.5	0.1	0.0903	0.9003	-0.0997	25.0	15.0	3.0	0.0071	-0.0493	-1.0493
1.0	0.2	0.1627	0.8024	-0.1976	30.0	10.0	2.0	0.1230	-0.0563	-1.0563
2.0	0.4	0.2610	0.6174	-0.3826	35.0	5.0	1.0	0.3096	0.1988	-0.8012
3.0	0.6	0.3099	0.4530	-0.5470	37.0	3.0	0.6	0.3099	0.4530	-0.5470
5.0	1.0	0.3096	0.1988	-0.8012	38.0	2.0	0.4	0.2610	0.6174	-0.3826
10.0	2.0	0.1230	-0.0563	-1.0563	39.0	1.0	0.2	0.1627	0.8024	-0.1976
15.0	3.0	0.0071	-0.0493	-1.0493	39.5	0.5	0.1	0.0903	0.9003	-0.0997
20.0	4.0	-0.0130	-0.0120	-1.0120	40.0	0	0	0	1.000	0

a. $\beta = 0.20$.

b. $\zeta(\beta x) = w/[(\alpha t a^2 \beta^2 d_0)/6(1-\nu)] = \zeta[\beta(L-x)]$.

c. $\theta(\beta x) - 1 = M_x/[(E\alpha t^2 d_0)/12(1-\nu)] = \theta[\beta(L-x)] - 1$.

where b_0 and b_1 are constants. It is desired that the deflections and extreme-fiber stresses be found, assuming that $L = 40$ in., $\beta = 0.2$, and $b_1/b_0 = 2$.

For the given temperature distribution, equations (38) yield

$$N_T = E\alpha t\,(b_0 + b_1 x)$$

and (70)

$$M_T = 0 \quad .$$

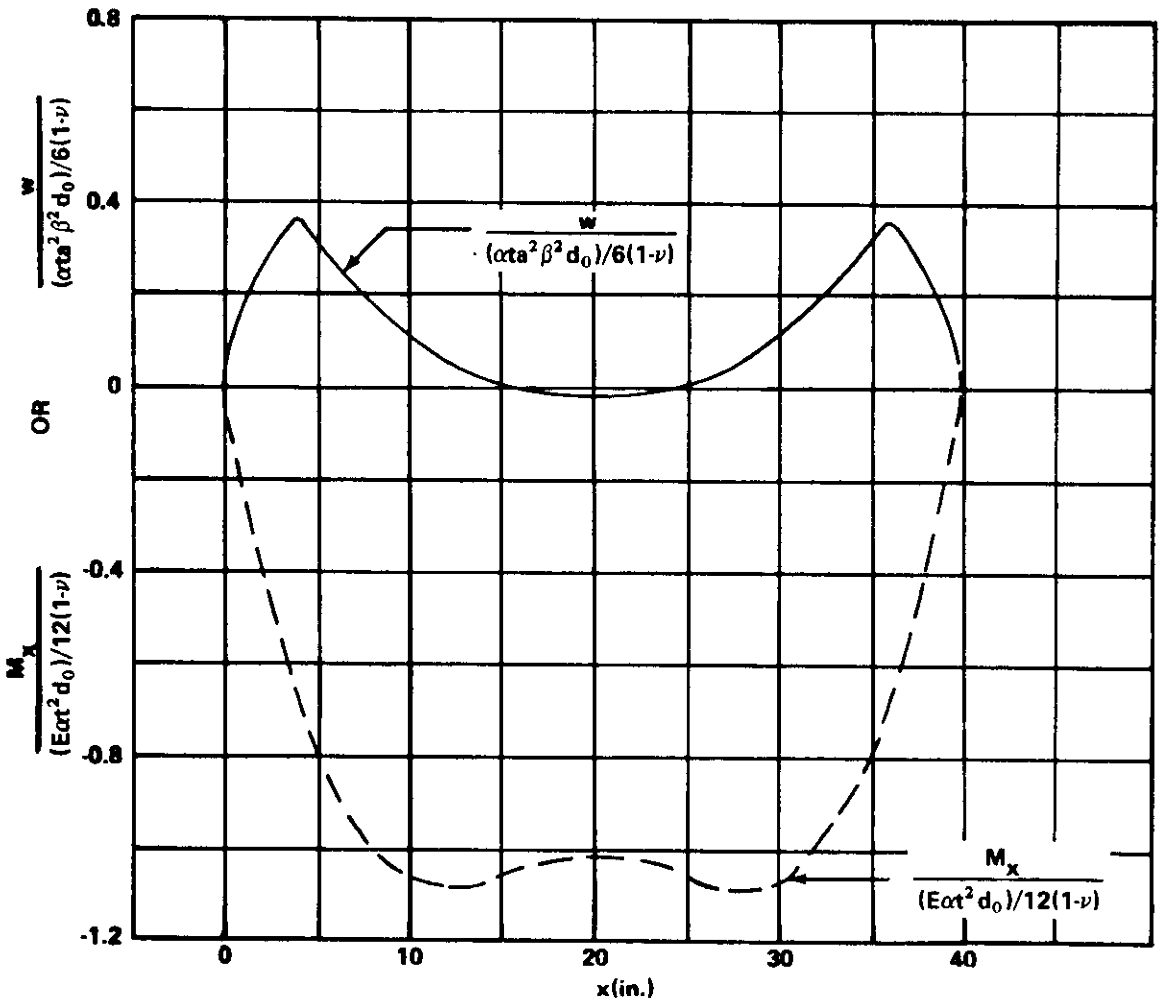


Figure 3.0-40. Nondimensional deflections and axial bending moments for example problem No. 1.

Then for $0 \le x \le L/2$, equation (4) becomes

$$w = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) - a\alpha (b_0 + b_1 x) \quad . \tag{71}$$

At $x = 0$, the clamped condition requires that

$$w = \frac{dw}{dx} = 0 \quad . \tag{72}$$

From equations (71) and (72), the constants C_1 and C_2 are found to be

$$C_1 = \alpha a b_0$$

and (73)

$$C_2 = \alpha a b_0 + \frac{\alpha a b_1}{\beta} \quad .$$

Hence, equation (71) may now be rewritten as follows:

$$w = e^{-\beta x}\left[\alpha a b_0(\cos \beta x + \sin \beta x) + \frac{\alpha a b_1}{\beta} \sin \beta x\right] - \alpha a(b_0 + b_1 x) \quad . \tag{74}$$

The functions $\phi(\beta x)$ and $\zeta(\beta x)$, tabulated by Timoshenko and Woinowsky-Krieger [11], may now be introduced to obtain

$$w = \alpha a b_0 \phi(\beta x) + \frac{\alpha a b_1}{\beta} \zeta(\beta x) - \alpha a(b_0 + b_1 x) \tag{75}$$

where

$$\phi(\beta x) = e^{-\beta x}(\cos \beta x + \sin \beta x) \tag{76}$$

and

$$\zeta(\beta x) = e^{-\beta x} \sin \beta x \quad . \tag{76 (Con.)}$$

By substituting equation (75) into (45), the bending moments can be expressed as

$$M_x = 2\alpha a b_0 D_b \beta^2 \left[\psi(\beta x) + \frac{b_1}{b_0 \beta} \theta(\beta x) \right]$$

and (77)

$$M_\phi = \nu M_x$$

where the functions $\psi(\beta x)$ and $\theta(\beta x)$ are tabulated by Timoshenko and Woinowsky-Krieger [11] and are defined as follows:

$$\psi(\beta x) = e^{-\beta x} (\cos \beta x - \sin \beta x)$$

and (78)

$$\theta(\beta x) = e^{-\beta x} \cos \beta x \quad .$$

The extreme-fiber stresses can then be determined from

$$\sigma_x = \mp \frac{6M_x}{t^2} . \tag{79}$$

and

$$\sigma_\phi = \frac{N_\phi}{t} \mp \frac{6M_\phi}{t^2} \qquad (79)\ (\text{Con.})$$

where the upper signs correspond to the outermost fibers. By substituting equations (75) and (70) into (47), the following expression is obtained for N_ϕ :

$$N_\phi = -E\alpha b_0 t \phi(\beta x) - \frac{E\alpha b_1 t}{\beta} \zeta(\beta x) \quad . \qquad (80)$$

For the other half of the cylinder $(L/2 \leq x \leq L)$, equation (40) must be used, which, in view of equations (70), becomes

$$w = e^{-\beta(L-x)} [C_3 \cos \beta(L-x) + C_4 \sin \beta(L-x)] - \alpha a(b_0 + b_1 x) \ . \qquad (81)$$

At $x = L$, the clamped condition requires that

$$w = \frac{dw}{dx} = 0 \quad . \qquad (82)$$

From equations (81) and (82), the constants C_3 and C_4 are found to be

$$C_3 = \alpha a(b_0 + b_1 L)$$

and (83)

$$C_4 = -\frac{\alpha a b_1}{\beta} + \alpha a(b_0 + b_1 L) \quad .$$

Then proceeding in the same manner as for the other half of the cylinder, the following expressions are found for the deflections, moments, and extreme-fiber stresses:

$$w = \alpha a b_0 \left(1 + \frac{b_1}{b_0} L\right) \phi[\beta(L - x)] - \frac{\alpha a b_1}{\beta} \zeta[\beta(L - x)] - \alpha a b_0 \left(1 + \frac{b_1}{b_0} x\right) , \tag{84}$$

$$M_x = 2 a \alpha \beta^2 b_0 D_b \left\{ \left(1 + \frac{b_1}{b_0} L\right) \psi[\beta(L - x)] - \frac{b_1}{b_0 \beta} \theta[\beta(L - x)] \right\} , \tag{85}$$

$$M_\phi = \nu M_x ,$$

$$\sigma_x = \mp \frac{6M_x}{t^2} ,$$

and (86)

$$\sigma_\phi = \frac{N_\phi}{t} \mp \frac{6M_\phi}{t^2}$$

where

$$N_\phi = -E\alpha b_0 t \left(1 + \frac{b_1}{b_0} L\right) \phi[\beta(L - x)] + \frac{E\alpha b_1 t}{\beta} \zeta[\beta(L - x)] . \tag{87}$$

Here again, the upper signs in the stress formulas correspond to the outermost fibers.

The foregoing relationships for the two halves of the cylinder were used to obtain the nondimensional solution shown in Table 3.0-8. These results are plotted in Figure 3.0-41.

TABLE 3.0-8. NONDIMENSIONAL TABULAR SOLUTION FOR w AND M_x (EXAMPLE NO. 2)

1	2	3	4	5	6	7	8	9
x	βx	$\phi(\beta x)$	$\frac{b_1}{\beta b_0}\zeta(\beta x)$	$-\left(1+\frac{b_1}{b_0}x\right)$	A_1[a]	$\psi(\beta x)$	$\frac{b_1}{b_0\beta}\theta(\beta x)$	A_2[b]
0	0	1.000	0	-1	0	1.0	10.0	11.0
0.5	0.1	0.9907	0.903	-2	-0.1063	0.81	9.003	9.813
1.0	0.2	0.9651	1.627	-3	-0.4079	0.6398	8.024	8.6638
2.0	0.4	0.8784	2.610	-5	-1.5116	0.3564	6.174	6.5304
3.0	0.6	0.7628	3.099	-7	-3.1382	0.1431	4.530	4.6731
5.0	1.0	0.5083	3.096	-11	-7.3957	-0.1108	1.988	1.8772
10.0	2.0	0.0667	1.230	-21	-19.7033	-0.1794	-0.563	-0.7424
15.0	3.0	-0.0423	0.071	-31	-30.9713	-0.0563	-0.493	-0.5493
20.0	4.0	-0.0258	-0.117	-41	-41.1428	0.0019	-0.120	-0.1181

a. $A_1 = (w/\alpha a b_0) = (3) + (4) + (5)$.

b. $A_2 = (M_x/2\alpha a b_0 D_b \beta^2) = (7) + (8)$.

x	(L-x)	β(L-x)	$\left(1+\frac{b_1}{b_0}L\right) \times \phi[\beta(L-x)]$	$-\frac{b_1}{b_0\beta} \times \zeta[\beta(L-x)]$	$-\left(1+\frac{b_1}{b_0}x\right)$	$\frac{w}{\alpha a b_0}$ = (4) + (5) + (6)	$\left(1+\frac{b_1}{b_0}L\right) \times \psi[\beta(L-x)]$	$\frac{b_1}{b_0\beta} \times \theta[\beta(L-x)]$	$\frac{M_x}{2\alpha a b_0 D_b \beta^2}$ = (8) - (9)
1	2	3	4	5	6	7	8	9	10
20	20	4	-2.0898	0.117	-41	-42.9728	0.1539	-0.120	0.2739
25	15	3	-3.4263	-0.071	-51	-54.4973	-4.5603	-0.493	-4.0673
30	10	2	5.4027	-1.230	-61	-56.8273	-14.5314	-0.563	-13.9684
35	5	1.0	41.1723	-3.096	-71	-32.9237	-8.9748	1.988	-10.9628
37	3	0.6	61.7869	-3.099	-75	-16.3121	11.5911	4.530	7.0611
38	2	0.4	71.1504	-2.610	-77	-8.4596	28.8684	6.174	22.6944
39	1.0	0.2	78.1731	-1.627	-79	-2.4539	51.8238	8.024	43.7998
39.5	0.5	0.1	80.2467	-0.903	-80	-0.6563	65.610	9.003	56.607
40	0	0	81.00	0	-81	0	81.0	10.00	71.00

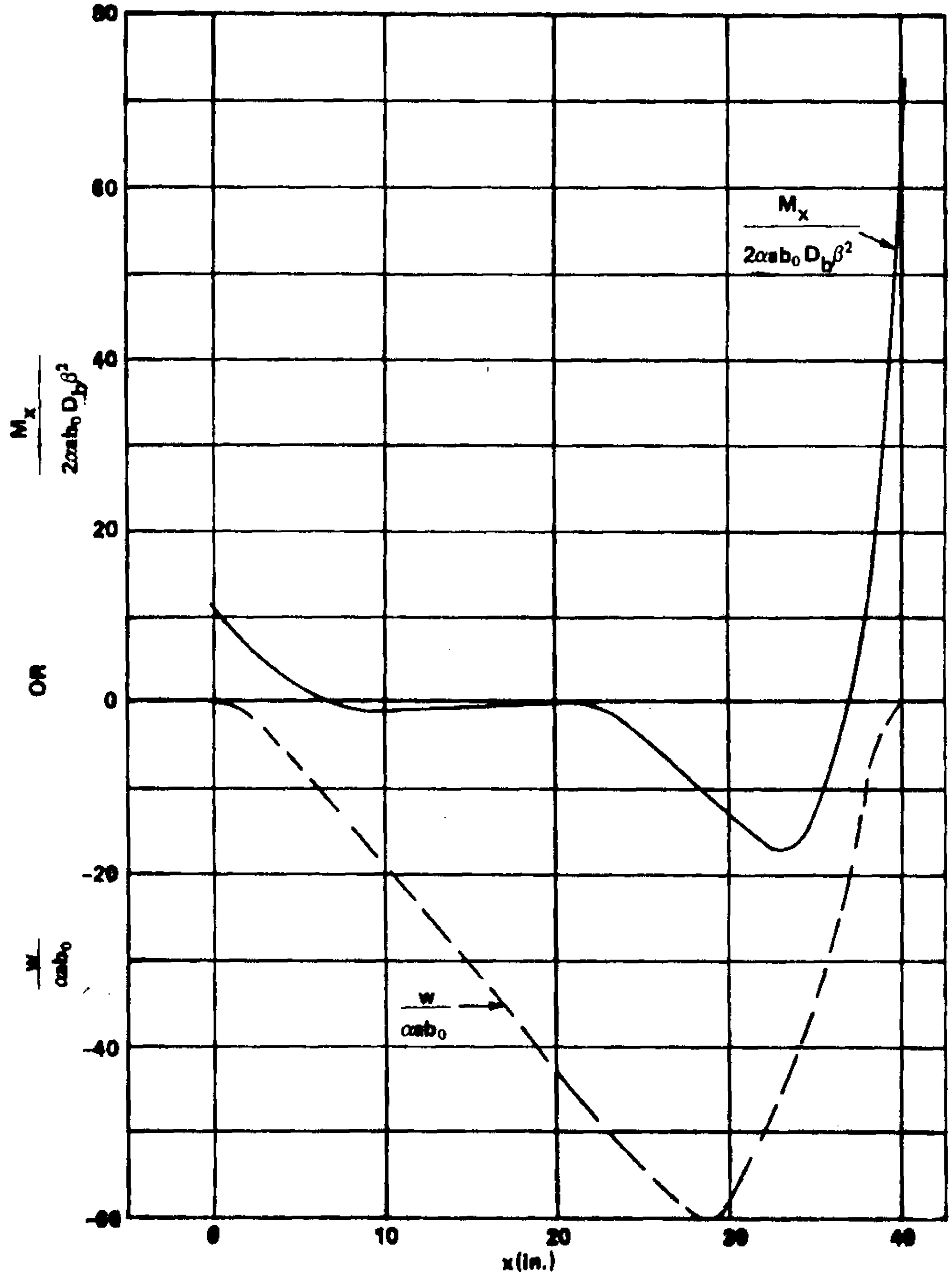


Figure 3.0-41. Nondimensional deflections and axial bending moments for example problem No. 2.

III. Thermal Stresses and Deflections - Constant Radial Gradient, Axisymmetric Axial Gradient.

A. Configuration.

The design curves and equations presented here apply only to thin-walled, right circular cylinders which are of constant thickness and are made of isotropic material. It is assumed that the shell wall is free of holes and that it obeys Hooke's law. The method is valid only when $\lambda \geq \pi$. Figure 3.0-42 depicts the isotropic cylinder shell configuration. Figure 3.0-43 shows the sign convention for forces, moments, and pressures.

B. Boundary Conditions.

The following types of boundary conditions are discussed:

1. Free edge; that is,

$$Q_x = M_x = 0 \quad . \tag{88}$$

2. Simply supported edge; that is,

$$w = M_x = 0 \quad . \tag{89}$$

3. Clamped edge; that is,

$$w = \frac{dw}{dx} = 0 \quad . \tag{90}$$

All possible combinations of these boundary conditions are permitted. Hence, it is not required that those at $x = 0$ be the same as those at $x = L$. However,

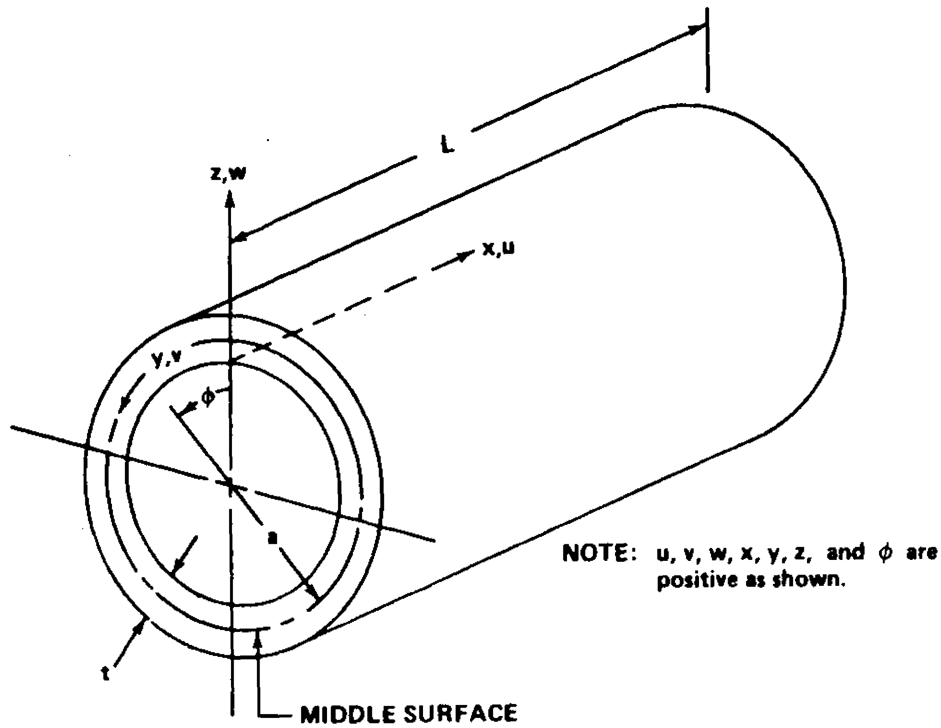


Figure 3.0-42. Isotropic cylindrical shell configuration for thermal stresses and deflections.

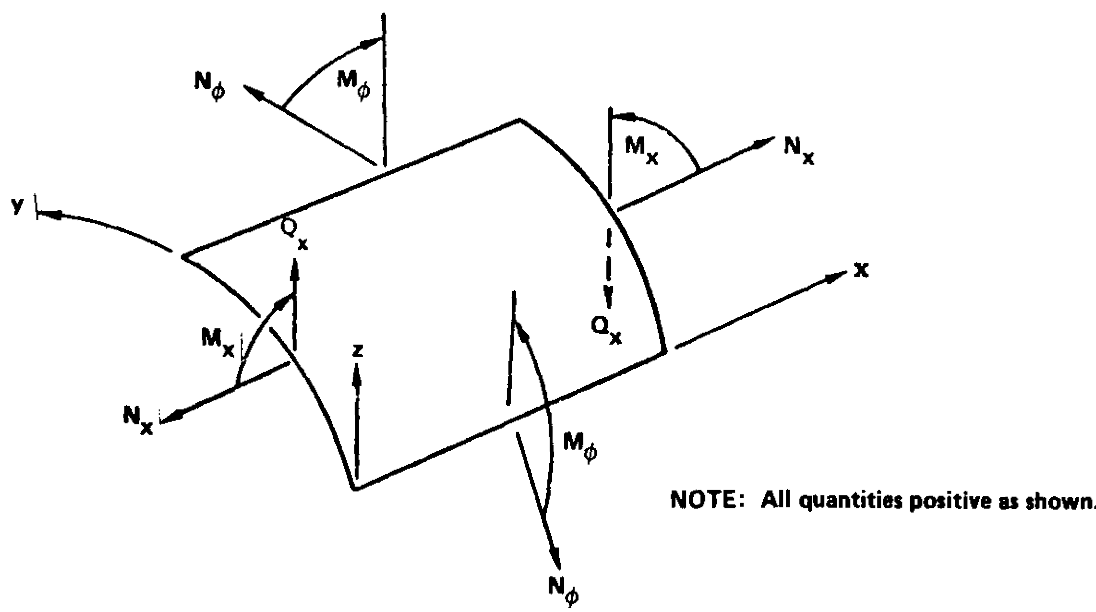


Figure 3.0-43. Sign convention for forces, moments, and pressures for thermal stresses and deflections.

in every case, it is assumed that the cylinder is unrestrained in the axial direction (N_x 0) .

C. Temperature Distribution.

The supposition is made that no temperature variations occur through the wall thickness. However, the cylinder may have any axisymmetric surface gradient for which the product αT can be adequately represented by a fifth-degree (or lower) polynomial. Therefore, subject to that restriction, the permissible distributions are of the form

$$\alpha T = \alpha(x)\, T(x) \quad . \tag{91}$$

D. Design Curves and Equations.

In Ref. 23, Newman and Forray present the practical method given below to compute the thermal stresses, deflections, and rotations in circular cylindrical shells which comply with the foregoing specifications. The primary relationships are expressed in series form and the necessary term-by-term coefficients can be obtained from Figures 3.0-44 through 3.0-49. As indicated by equation (91), the product αT will be a function of x and, in order to apply this method, this function must first be approximated by the polynomial

$$\alpha T = d_0 + d_1 \xi + d_2 \xi^2 + \ldots d_z \xi^Z = \sum_{k=0}^{Z} d_k \xi^k \tag{92}$$

where ξ is a dimensionless axial coordinate defined by the relationship

$$\xi = \frac{x}{L} \quad . \tag{93}$$

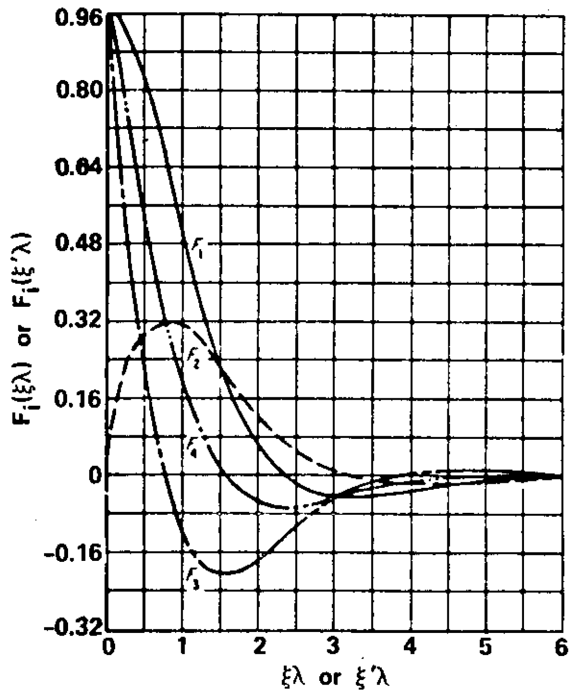


Figure 3.0-44. Functions F_i .

For the purposes of the technique given here, the following inequality must be satisfied:

$$Z \leq 5 \quad . \tag{94}$$

After the coefficients d_k have been established, the thermal stresses and distortions can be determined by using equations (96) through (98) in conjunction with the design curves. The constants A_1 and A_2 are based on the boundary conditions at $x = 0$ ($\xi = 0$) while A_3 and A_4 depend on the boundary conditions at $x = L$ ($\xi = 1$) . The formulas for these four values are listed in Table 3.0-9.

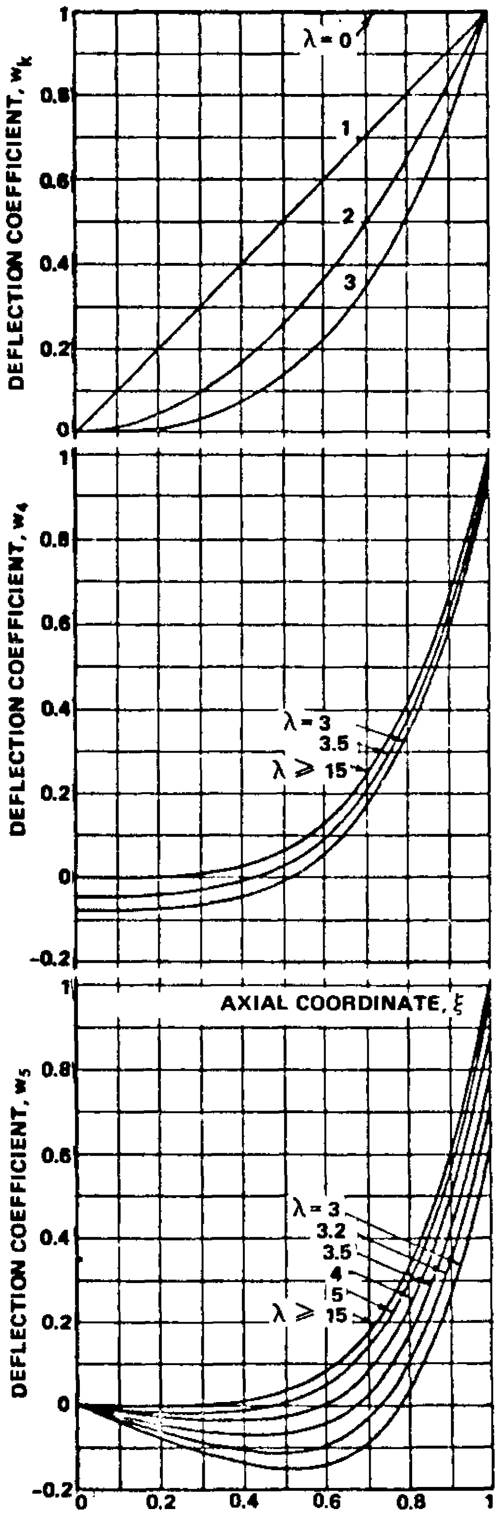


Figure 3.0-45. Deflection coefficients and k = 5 , the coefficient depends on λ .

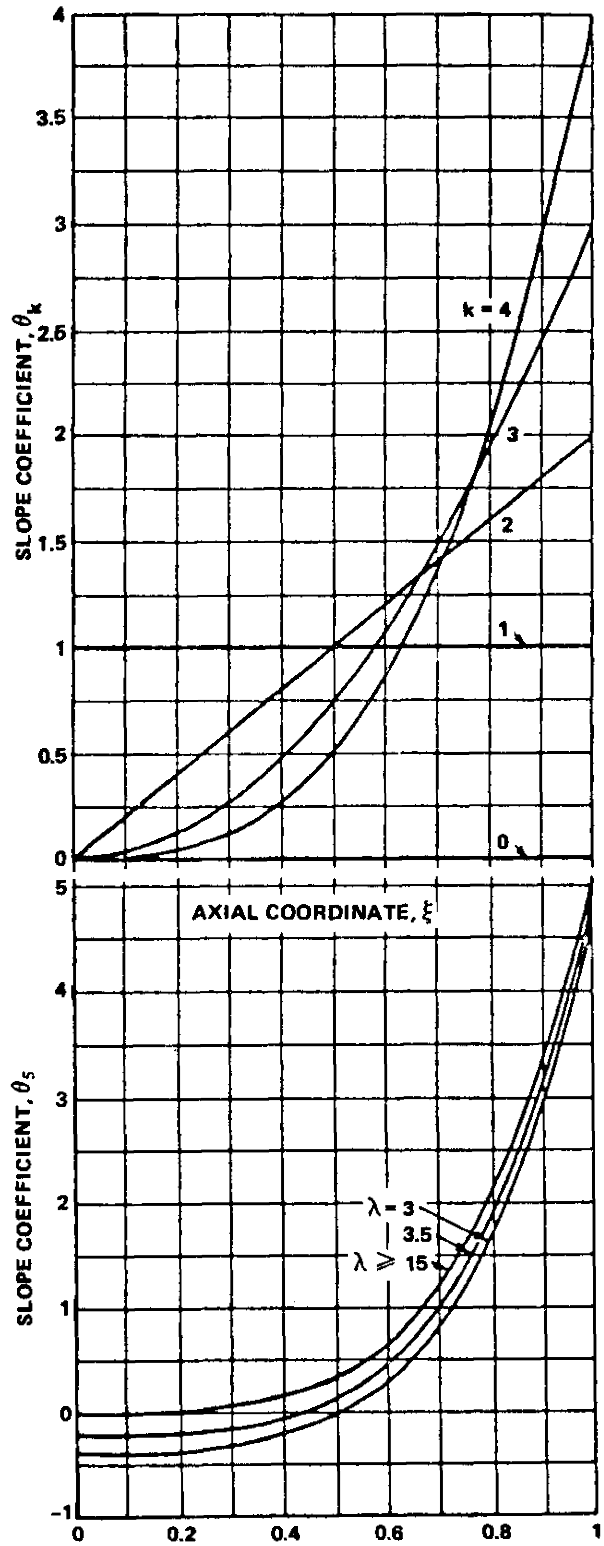


Figure 3.0-46. Slope coefficients (for k = 5 , the coefficient depends on λ) .

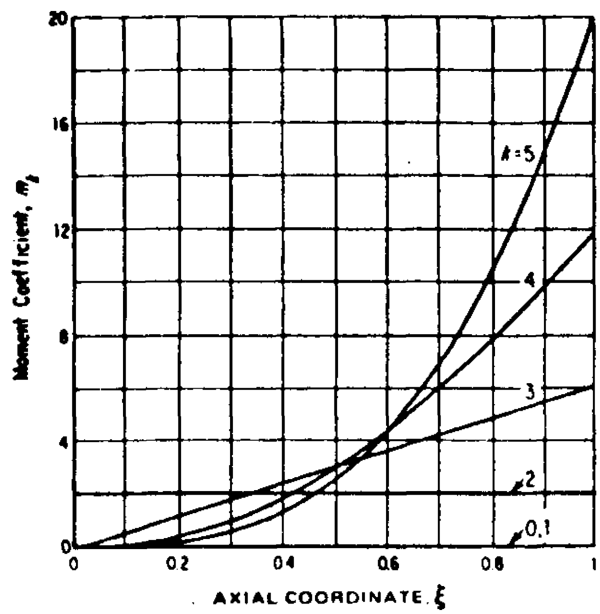


Figure 3.0-47. Moment coefficients.

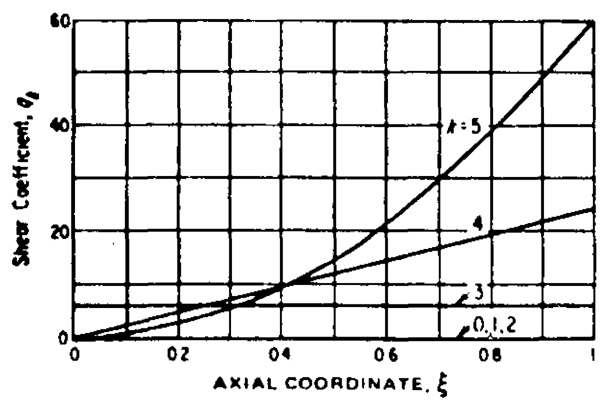


Figure 3.0-48. Shear coefficients.

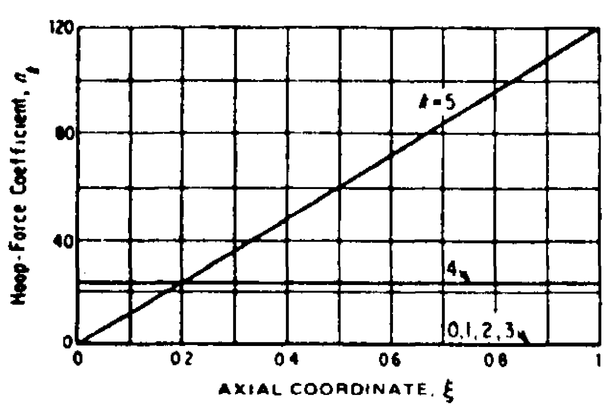


Figure 3.0-49. Hoop-force coefficients.

TABLE 3.0-9. FORMULAS FOR THE CONSTANTS A_1 THROUGH A_4

Cylinder End	Boundary Condition	Constants A_i
$\xi = 0$ $(x = 0)$	Free	$A_1 = \frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k m_k(0)$ $A_2 = -\frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k \left[m_k(0) + \frac{1}{\lambda} q_k(0) \right]$
$\xi = 0$ $(x = 0)$	Simple support	$A_1 = \frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k m_k(0)$ $A_2 = - \sum_{k=0}^{Z} d_k w_k(0)$
$\xi = 0$ $(x = 0)$	Clamped	$A_1 = - \sum_{k=0}^{Z} d_k \left[w_k(0) + \frac{1}{\lambda} \theta_k(0) \right]$ $A_2 = - \sum_{k=0}^{Z} d_k w_k(0)$
$\xi = 1$ $(x = L)$	Free	$A_3 = \frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k m_k(1)$ $A_4 = \frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k \left[\frac{1}{\lambda} q_k(1) - m_k(1) \right]$
$\xi = 1$ $(x = L)$	Simple support	$A_3 = \frac{1}{2\lambda^2} \sum_{k=0}^{Z} d_k m_k(1)$ $A_4 = - \sum_{k=0}^{Z} d_k w_k(1)$
$\xi = 1$ $(x = L)$	Clamped	$A_3 = \sum_{k=0}^{Z} d_k \left[\frac{1}{\lambda} \theta_k(1) - w_k(1) \right]$ $A_4 = - \sum_{k=0}^{Z} d_k w_k(1)$

The solutions are based on classical small-deflection shell theory. Therefore, it is important for the user to be aware of this when the method is applied to pressurized cylinders by superimposing the thermal stresses and deformations upon the corresponding values due solely to pressure; that is, because of the dependence upon classical theory, the method presented here cannot account for nonlinear coupling between thermal deflections and the pressure-related meridional loads.

In addition, it is assumed that Young's modulus and Poisson's ratio are unaffected by temperature changes. Hence, the user must select single effective values for each of these properties by employing some type of averaging technique.

E. Summary of Equations and Nondimensional Coefficients.

$$\alpha T = d_0 + d_1\xi + d_2\xi^2 + \ldots d_z\xi^z = \sum_{k=0}^{Z} d_k\xi^k \quad , \tag{95}$$

$$\frac{w}{a} = A_1F_2(\xi\lambda) + A_2F_4(\xi\lambda) + A_3F_2(\xi'\lambda) + A_4F_4(\xi'\lambda) + \sum_{k=0}^{Z} d_k w_k \quad ,$$

$$\frac{L}{a}\,\theta = \lambda[A_1F_3(\xi\lambda) - A_2F_1(\xi\lambda) - A_3F_3(\xi'\lambda) + A_4F_1(\xi'\lambda)] + \sum_{k=0}^{Z} d_k\theta_k \quad ,$$

$$\frac{L^2M_x}{aD_b} = 2\lambda^2[-A_1F_4(\xi\lambda) + A_2F_2(\xi\lambda) - A_3F_4(\xi'\lambda) + A_4F_2(\xi'\lambda)] + \sum_{k=0}^{Z} d_k m_k \ ,$$

$$M_\phi = \nu M_x \quad , \tag{96}$$

$$\frac{L^3 Q_x}{a D_b} = 2\lambda^3 [A_1 F_1(\xi\lambda) + A_2 F_3(\xi\lambda) - A_3 F_1(\xi'\lambda) - A_4 F_3(\xi'\lambda)] + \sum_{k=0}^{Z} d_k q_k \quad ,$$

and

$$\frac{L^4 N_\phi}{a^2 D_b} = 4\lambda^4 [A_1 F_2(\xi\lambda) + A_2 F_4(\xi\lambda) + A_3 F_2(\xi'\lambda) + A_4 F_4(\xi'\lambda)] - \sum_{k=0}^{Z} d_k n_k \quad ,$$

where

$$\lambda = L \frac{\sqrt[4]{3(1 - \nu^2)}}{(at)^{1/2}} \quad ,$$

$$D_b = \frac{Et^3}{12(1 - \nu^2)} \quad , \tag{97}$$

$$\xi = \frac{x}{L} \quad ,$$

and

$$\xi' = 1 - \xi \quad .$$

The stresses at any location are given by the following:

$$\sigma_x = -\frac{12z}{t^3} M_x \tag{98}$$

and

$$\sigma_\phi = \frac{N_\phi}{t} - \frac{12z}{t^3} M_\phi \quad . \tag{98 (Con.)}$$

3.0.8.2 Isotropic Conical Shells.

This section concerns the thermostructural analysis of thin-walled, right circular, isotropic conical shells. The organization here is somewhat different from that of previous sections which cover isotropic flat plates. This is due to certain fundamental differences between the physical behavior of flat plates and shells. Flat-plate deformations are of such a nature that it is helpful to group the solutions for stable constructions into the following categories:

1. Temperature gradients through the thickness
2. Uniform temperatures through the thickness.

Except for the special case of self-equilibrating gradients through the thickness $(N_T = M_T = 0)$, the first of these two cases involves out-of-plane bending which is, of course, accompanied by displacements normal to the middle surface of the undeformed plate. In case 2, the plate remains flat; that is, the only displacements occur in directions parallel to the original middle surface and no out-of-plane bending occurs. The indicated separation of cases is therefore a logical format for the sections dealing with flat plates. However, the situation is not the same for shell structures. For these components, there is no need to isolate the foregoing types of thermal conditions.

This is because either type of temperature distribution, in conjunction with clamped or simply supported boundaries, will lead to both membrane loading and bending about the shell-wall middle surface. Consequently, for stable shell constructions which comply with either case 1 or 2, the analysis methods are given as follows as a single grouping.

Configuration.

The design equations provided here apply only to long $(L \geq 2\pi/\lambda B)$, thin-walled, truncated, right circular cones which are of constant thickness, are made of isotropic material, and satisfy the inequality

$$x_A > 3t \cot \phi \quad . \tag{99}$$

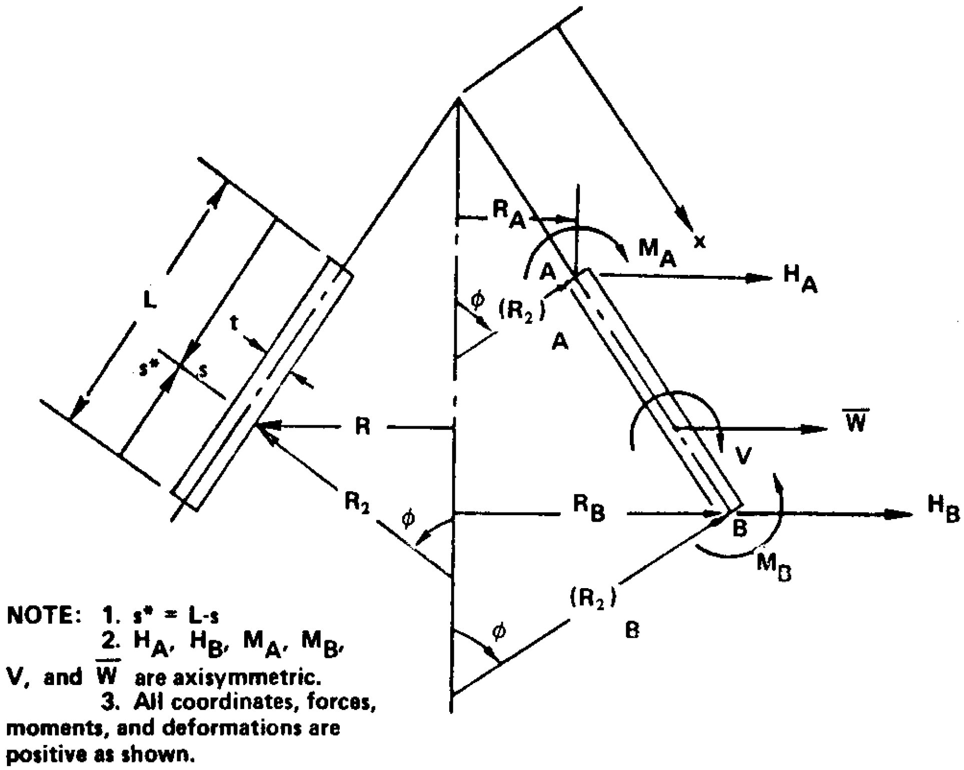
It is assumed that the shell wall is free of holes and obeys Hooke's law. Figure 3.0-50 depicts the subject configuration, as well as most of the notation and sign conventions of interest.

Boundary Conditions.

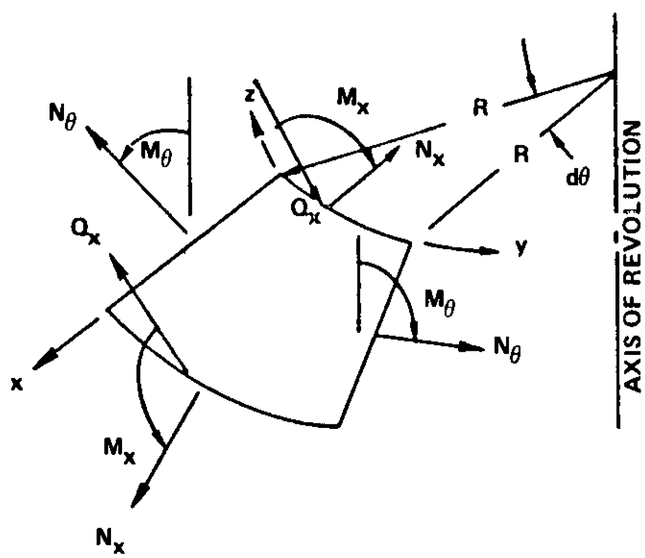
The method presented here can be applied where any of the following boundary conditions are present:

1. Free edges
2. Simply supported edges
3. Clamped edges.

All possible combinations of these boundaries are permitted; that is, it is not required that those at x_A be the same as those at x_B. However, in every case, it is assumed that the cone is unrestrained in the axial direction.



a. Overall truncated cone.



b. Positive directions for the stress resultants and coordinates.

Figure 3.0-50. Configuration, notation, and sign convention for conical shell.

Temperature Distribution.

The following types of temperature distributions may be present:

1. A linear gradient through the wall thickness subject to the provision that the temperature change T need not vanish at the middle surface.

2. Axisymmetric meridional gradients.

The permissible distributions can therefore be expressed in the form

$$T = T_1(s) + T_2(s) \frac{z}{t} \quad . \tag{100}$$

Naturally, any of the special cases for this equation are applicable; that is, either or both of $T_1(s)$ and $T_2(s)$ can be finite constants and either may be equal to zero.

Design Equations.

A number of methods for solving the subject problem have been published, including those of Refs. 24 through 27. In the approach presented here, particular solutions to the governing differential equations are found in the manner suggested by Tsui [16]. As in Refs. 25 and 28, the complementary solutions are obtained by an equivalent-cylinder approximation. When greater accuracy is desired, the exact complementary solutions published by Johns and Orange [29] may be used.

Throughout this section it is assumed that Young's modulus and Poisson's ratio are unaffected by temperature changes. Hence, the user must select single effective values for each of these properties by employing some type of averaging technique. The same approach may be taken with regard to the coefficient of thermal expansion. On the other hand, the temperature-dependence of this property may be accounted for by recognizing that it is the product αT which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α. When this approach is taken, any mention of a linear temperature distribution is actually making reference to a straight-line variation of the product αT.

In addition, the method outlined here is based on classical small-deflection theory. It is important to keep this in mind when applying to pressurized cones by superimposing the thermal stresses and deformations upon the corresponding values due solely to pressure; that is, because of the dependence upon classical theory, the method of this manual cannot account for nonlinear coupling between thermal deflections and pressure-related meridional loads.

The governing differential equations for the subject cone are given by Tsui [16] as follows:

$$L'(U) - VEt \tan\phi = -x \frac{dN_T}{dx}$$

and (101)

$$L'(V) + U \frac{1}{D_b} \cot\phi = -\frac{1}{D_b} \frac{\cot\phi}{(1-\nu)} \frac{dM_T}{dx}$$

where

$$D_b = \frac{Et^3}{12(1-\nu^2)} \quad ,$$

$$M_T = E\alpha \int_{-t/2}^{t/2} Tz\, dz \quad ,$$

$$N_T = E\alpha \int_{-t/2}^{t/2} T\, dz \quad , \tag{102}$$

and

$$U = xQ_x$$

and L' is the operator,

$$L'(\ ) = \cot\phi \left[x \frac{d^2(\ )}{dx^2} + \frac{d(\ )}{dx} - \frac{(\ )}{x} \right] \quad . \tag{103}$$

To obtain the desired solution, a three-step procedure is employed as outlined below:

Step 1. Find a particular solution to equations (101).

Step 2. Find a solution to the homogeneous equations,

$$L'(U) - VEt \tan\phi = 0$$

and (104)

$$L'(V) + U \frac{1}{D_b} \cot\phi = 0 \quad ,$$

such that superposition of these results upon those of Step 1 satisfies the boundary conditions which can be expressed as follows:

$$\text{Free edge:} \qquad Q_x = M_x = 0 \ .$$

$$\text{Simply supported edge:} \qquad \overline{W} = M_x = 0 \ . \tag{105}$$

$$\text{Clamped edge:} \qquad \overline{W} = V = 0 \ .$$

The results from this step are referred to as the complementary solution. Note that equations (104) are obtained by setting the right-hand sides of equations (101) equal to zero.

Step 3. Superimpose the particular and complementary solutions. To accomplish the first of these steps, the functions N_T and M_T are first approximated as polynomials. It is then assumed that the particular solutions U^P and V^P can be expressed in the form

$$U^P = C_{-1}x^{-1} + C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$$

and (106)

$$V^P = d_{-1}x^{-1} + d_0 + d_1x + d_2x^2 + d_3x^3 + \ldots + d_nx^n$$

where n is an integer whose value is a function of the polynomial degree required for a sufficiently accurate representation of N_T and M_T. If these formulations for N_T, M_T, U^P, and V^P are substituted into equations (101) and like powers of x are equated, a system of simultaneous equations is obtained where the unknowns are the various polynomial coefficients. These equations can be solved for $C_{(\ )}$ and $d_{(\ )}$ and hence U^P and V^P. The associated radial deflection and stress resultants of interest can then be determined from

$$\overline{W}^P = \frac{\cos^2\phi}{Et\sin\phi}\left(x\frac{dU^P}{dx} - \nu U^P\right) + \alpha R T_m \quad ,$$

$$Q_x^P = \frac{U^P}{x} \quad ,$$

$$N_x^P = Q_x \cot\phi \quad , \qquad (107)$$

$$N_\theta^P = \frac{d}{dx}(R_2 Q_x) = \cot\phi\,\frac{dU^P}{dx} \quad ,$$

$$M_x^P = D_b\left(\frac{dV^P}{dx} + \frac{\nu}{R_2}V^P\cot\phi\right) - \frac{M_T}{(1-\nu)} \quad ,$$

and

$$M_\theta^P = D_b\left(\frac{1}{R_2}V^P\cot\phi + \nu\frac{dV^P}{dx}\right) - \frac{M_T}{(1-\nu)}$$

where

$$T_m = \frac{1}{t}\int_{-t/2}^{t/2} T\,dz \quad . \qquad (108)$$

The complementary solutions corresponding to the edge-loaded cone of Figure 3.0-51 are given as equations (109) and (110). Those corresponding to the edge-loaded cone of Figure 3.0-52 are given as equations (111) and (112).

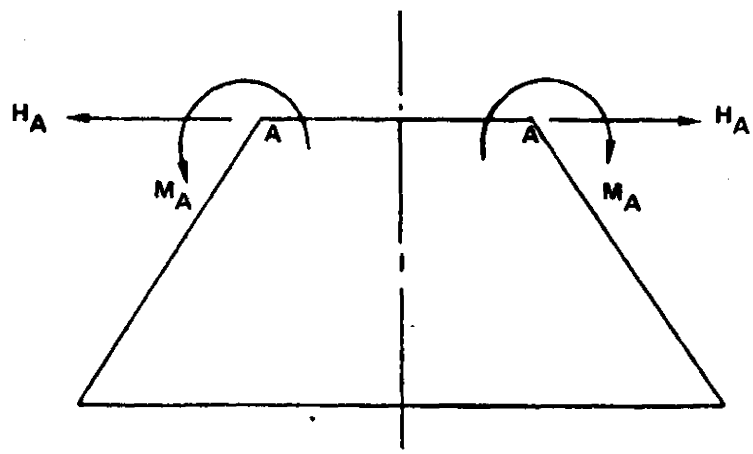


Figure 3.0-51. Truncated cone edge-loaded at top.

$$\bar{W}_A^C = \frac{\sin\phi}{2\lambda_A^3 D_b}(\lambda_A M_A + H_A \sin\phi) \quad ,$$

$$V_A^C = \frac{1}{2\lambda_A^2 D_b}(2\lambda_A M_A + H_A \sin\phi) \quad ,$$

$$\bar{W}^C = \frac{\sin\phi}{2\lambda_A^3 D_b}[\lambda_A M_A \bar{\psi}(\lambda_A s) + H_A (\sin\phi)\, \bar{\theta}(\lambda_A s)] \quad ,$$

$$V^C = \frac{1}{2\lambda_A^2 D_b}[2\lambda_A M_A \bar{\theta}(\lambda_A s) + H_A (\sin\phi)\, \bar{\varphi}(\lambda_A s)] \quad , \qquad (109)$$

$$Q_x^C = [2\lambda_A M_A \bar{\zeta}(\lambda_A s) - H_A (\sin\phi)\, \bar{\psi}(\lambda_A s)] \quad ,$$

$$N_x^C = Q_x \cot\phi \quad ,$$

$$N_\theta^C = \frac{\overline{W}Et}{R} + \nu N_x \quad ,$$

(109)
(Con.)

$$M_x^C = -\frac{1}{2\lambda_A}\left[2\lambda_A M_A \overline{\phi}(\lambda_A s) + 2H_A(\sin\phi)\,\overline{\zeta}(\lambda_A s)\right] \quad ,$$

and

$$M_\theta^C = \nu M_x$$

where

$$\lambda_A = \sqrt[4]{\frac{3(1-\nu^2)}{(R_2)_A^2\, t^2}} \quad ,$$

$$(R_2)_A = \frac{R_A}{\sin\phi} \quad , \qquad (110)$$

and

$$D_b = \frac{Et^3}{12(1-\nu^2)}$$

and $\overline{\phi}$, $\overline{\psi}$, $\overline{\theta}$, and $\overline{\zeta}$ are the functions ϕ, ψ, θ, and ζ, respectively, which are tabulated on pages 472-473 of Ref. 11.

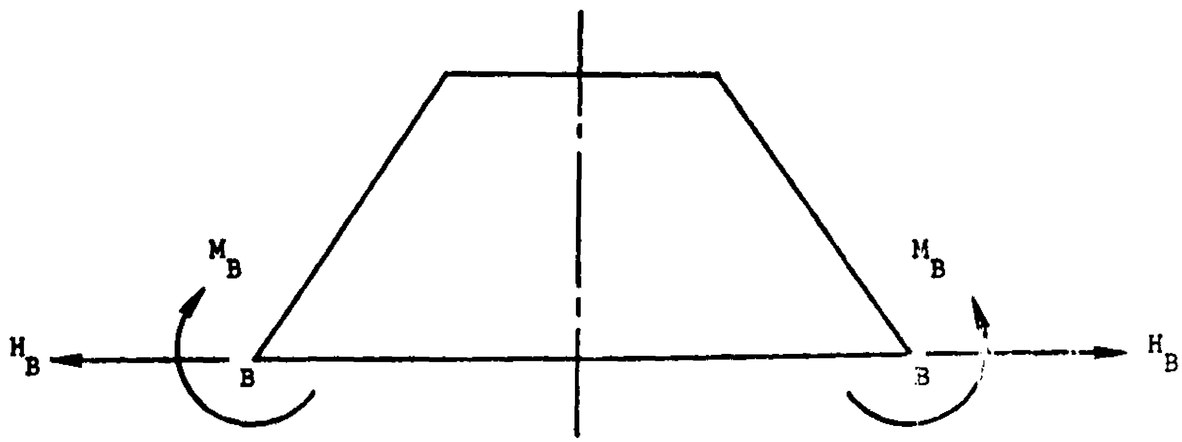


Figure 3.0-52. Truncated cone edge-loaded at bottom.

$$\overline{W}_B^C = \frac{\sin\phi}{2\lambda_B^3 D_b} (\lambda_B M_B + H_B \sin\phi) \quad ,$$

$$V_B^C = -\frac{1}{2\lambda_B^2 D_b} (2\lambda_B M_B + H_B \sin\phi) \quad ,$$

$$\overline{W}^C = \frac{\sin\phi}{2\lambda_B^3 D_b} [\lambda_B M_B \overline{\psi}(\lambda_B s^*) + H_B (\sin\phi)\ \overline{\theta}(\lambda_B s^*)] \quad ,$$

$$V^C = -\frac{1}{2\lambda_B^2 D_b} [2\lambda_B M_B \overline{\theta}(\lambda_B s^*) + H_B (\sin\phi)\ \overline{\psi}(\lambda_B s^*)] \quad ,$$

$$Q_x^C = -[2\lambda_B M_B \overline{\zeta}(\lambda_B s^*) - H_B (\sin\phi)\ \overline{\psi}(\lambda_B s^*)] \quad ,$$

$$N_x^C = Q_x \cot\phi \quad ,$$

$$N_\theta^C = \frac{\overline{W}Et}{R} + \nu N_x \quad , \qquad (111)$$

$$M_x^C = -\frac{1}{2\lambda_B}\left[2\lambda_B M_B \bar{\phi}(\lambda_B s^*) + 2 H_B (\sin\phi)\, \bar{\zeta}(\lambda_B s^*)\right] \quad ,$$

and

$$M_\theta^C = \nu M_x \tag{111 (Con.)}$$

where

$$\lambda_B = \sqrt[4]{\frac{3(1-\nu^2)}{(R_2)^2_B t^2}} \quad ,$$

$$(R_2)_B = \frac{R_B}{\sin\phi} \quad , \tag{112}$$

and

$$D_b = \frac{Et^3}{12(1-\nu^2)}$$

and $\bar{\phi}$, $\bar{\psi}$, $\bar{\theta}$, and $\bar{\zeta}$ are the functions ϕ, ψ, θ, and ζ, respectively, which are tabulated on pages 472-473 of Ref. 11.

After the particular and complementary solutions have been superimposed, the final thermal stresses can be computed from the following formulas:

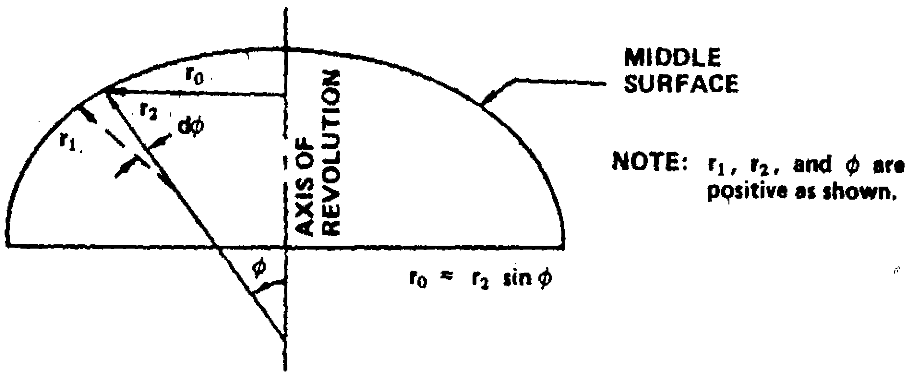
$$\sigma_x = \frac{12z}{t^3} M_x + \frac{N_x}{t} \quad \text{and} \quad \sigma_\theta = \frac{12z}{t^3} M_\theta + \frac{N_\theta}{t} \quad . \tag{113}$$

3.0.8.3 Isotropic Shells of Revolution of Arbitrary Shape.

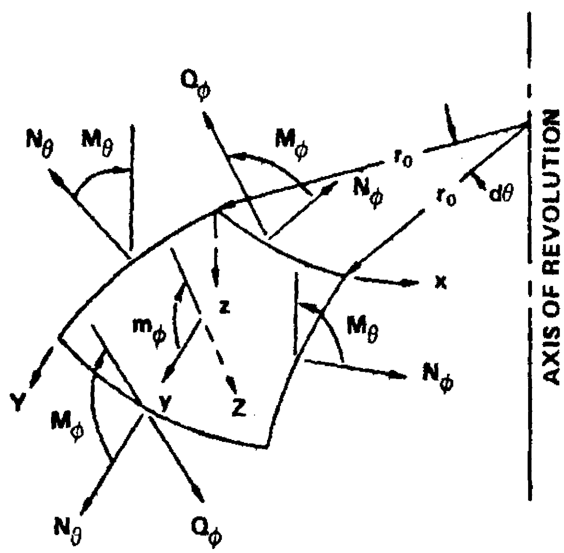
The discussion presented here is concerned with approximate small-deflection solutions for thin-walled shells of revolution having otherwise arbitrary shapes and made of isotropic material. A typical configuration, along with pertinent notation and sign conventions, is shown in Figure 3.0-53. It is assumed that the shell wall is free of holes and obeys Hooke's law. The temperature distribution must be axisymmetric but arbitrary gradients may be present both through the wall thickness and in the meridional direction. To determine the thermal stresses and deformations for the structures under discussion, the following sets of equations are available:

1. Equilibrium equations
2. Strain-displacement relationships
3. Stress-strain relationships.

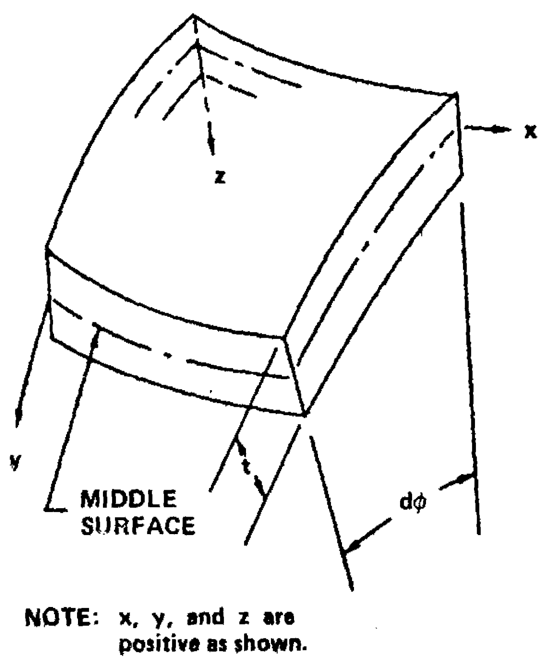
In principle, together with prescribed boundary conditions, these formulations should provide a sufficient basis for the development of closed-form, small-deflection solutions to the subject problems. However, it will often be extremely difficult, if not impossible, to achieve such solutions. Therefore, numerical integration procedures in conjunction with a digital computer program are frequently used to achieve the desired solution. On the other hand, still another approach may be taken by using approximations such as those cited by Fitzgerald in Ref. 30 or Christensen in Ref. 31. Since these approximations avoid the need for sophisticated mathematical and/or numerical



a. Overall shell of revolution.



b. Positive directions for forces, moments, pressures, and coordinates.



c. Element of shell wall.

Figure 3.0-53. Configuration, notation, and sign convention for arbitrary shell of revolution.

operations, they are well suited to a manual of this type. It would therefore be desirable to prepare a section which outlines detailed procedures along these lines. However, from a brief study of Refs. 30 and 31, it was concluded that they should be thoroughly explored before specific recommendations are made. Consequently, in the following paragraphs only the related broad concepts are presented.

The method of Ref. 30 relies heavily on the following set of equilibrium equations, which, except for the term involving m_ϕ , are derived in Ref. 11:

$$\frac{d}{d\phi}(N_\phi r_0) - N_\theta r_1 \cos\phi - r_0 Q_\phi + r_0 r_1 Y = 0 \quad ,$$

$$N_\phi r_0 + N_\theta r_1 \sin\phi + \frac{d}{d\phi}(Q_\phi r_0) + Z r_1 r_0 = 0 \quad , \tag{114}$$

and

$$\frac{d}{d\phi}(M_\phi r_0) - M_\theta r_1 \cos\phi - Q_\phi r_1 r_0 + m_\phi r_1 r_0 = 0 \quad .$$

These expressions are used in the following manner:

1. First the assumption is made that membrane forces

$$N_\theta = N_\phi = N_R \tag{115}$$

and bending moments

$$M_\theta = M_\phi = M_R \tag{116}$$

are present which completely arrest all thermal displacements.

These forces and moments simply furnish a starting point for the computations and do not represent the actual values which will be determined later in the procedure.

It follows that

$$N_R = -\frac{E}{(1-\nu)} \int_{-t/2}^{t/2} \alpha T \, dz \tag{117}$$

and

$$M_R = -\frac{E}{(1-\nu)} \int_{-t/2}^{t/2} \alpha T z \, dz \quad . \tag{118}$$

2. In general, the above type of force and moment distribution will not be in equilibrium unless one or more of the following is applied:

$$Q_\phi = (Q_\phi)_R \quad ,$$

$$Y = Y_R \quad , \tag{119}$$

$$Z = Z_R \quad ,$$

and

$$m_\phi = m_r \quad .$$

At this point, in order to achieve an approximate solution, Fitzgerald [30] makes the assumption that

$$Q_\phi = (Q_\phi)_R = 0 \tag{120}$$

and justifies this practice by performing an order-of-magnitude study of the error introduced. Then, proceeding with the analysis, equations (120) and (115) through (118) are substituted into the equilibrium relationships (114) to arrive at simple formulas for Y_R, Z_R, and m_R.

3. Recognizing that the actual shell is free of any of the above types of loading, it is necessary to restore the structure to this state by application of the following:

$$-Y_R;\ -Z_R;\ -m_R\quad .$$

This is done in a two-step procedure as outlined below.

4. The expressions

$$Y = -Y_R$$

and (121)

$$Z = -Z_R$$

are inserted into the first two of the equilibrium equations (114) while the assumption that

$$Q_\phi = 0 \tag{122}$$

is retained. The resulting equations are then solved for N_θ and N_ϕ. From the stress-strain relationships, the corresponding strains can be determined. After this, the strain-displacement formulations may be used to express the related rotations and deflections of the shell wall in terms of N_θ and N_ϕ. The bending moments M_θ and M_ϕ can then be established from the equations

$$M_\theta = - D_b(\chi_\theta + \nu \chi_\phi)$$

and (123)

$$M_\phi = - D_b(\chi_\phi + \nu \chi_\theta)$$

where

$$D_b = \frac{Et^3}{12(1 - \nu^2)} \quad , \tag{124}$$

while χ_θ and χ_ϕ are the curvature changes of the hoop and meridional fibers, respectively.

5. One may now proceed to substitute

$$m_\phi = - m_R \tag{125}$$

into the third of the equilibrium equations (114), along with the assumption that

$$M_\theta = M_\phi = 0 \quad . \tag{126}$$

Simple transformation then yields a formula for Q_ϕ which, together with equations (126) and the first two of (114), leads to simple expressions for N_θ and N_ϕ in terms of M_R. The use of equations (126) in this phase of the development is justified by Fitzgerald [30] on the basis of an error-magnitude study. Using the stress-strain and strain-displacement relationships, practical formulations can be derived for the rotations and displacements associated with the membrane loads N_θ and N_ϕ obtained in this step.

6. The final approximate values for the membrane loads, bending moments, rotations, and displacements are found by superposition of appropriate values from steps 1, 4, and 5. The stresses due to these membrane loads and bending moments must be augmented by those stresses associated with any self-equilibrating temperature distributions which exist through the thickness.

To focus attention on the general concepts involved in Fitzgerald's [30] approach, no mention is made in the foregoing steps of the need to satisfy prescribed boundary conditions in the problem solution. Therefore, it might now be helpful to note that, for this method, it is probably best first to obtain results under the assumption that no external constraints are present. Following this, edge forces and/or moments may be superimposed which enforce the required conditions at the boundaries.

The general philosophy behind the approach of Christensen [31] is very similar to that of Fitzgerald, although the details are quite different.

Christensen also relies heavily upon the equilibrium equations (114) but, for pure thermostructural problems, he makes no use of the loadings Y , Z , and m_ϕ . Hence, these quantities are taken equal to zero throughout the entire analysis, which is performed in the following manner:

1. First the assumption is made that

$$M_\theta = M_\phi = M_R \tag{127}$$

where

$$M_R = -\frac{E}{(1-\nu)} \int_{-t/2}^{t/2} \alpha T z \, dz \quad . \tag{128}$$

Here again, these moments simply furnish a starting point for the computations and do not represent the actual values which will be determined later in the procedure. These moments are inserted into the equilibrium equations (114). The third of these equations is then combined with the other two and two equations in the unknowns N_θ and N_ϕ are obtained.

2. By using the stress-strain and strain-displacement relationships, the two equations from step 1 are rewritten in terms of the temperature distribution and the middle-surface displacements v and w , where v is measured in the meridional direction and w is taken normal to the middle surface.

3. The two equations from step 2 are combined to arrive at a single formulation in terms of v and the temperature distribution.

4. The equation from step 3 is then solved subject to the boundary conditions at the shell apex. This is accomplished by assuming that v can be expressed as a polynomial and then calling upon the method of undetermined coefficients. The resulting expression for v must then be substituted into the appropriate equation from step 2 to obtain a solution for the displacement w .

5. From Timoshenko [11], the bending moments M_θ and M_ϕ which are associated with the displacements v and w can be determined. Christensen [31] refers to these as corrective moments and, if they are not small with respect to M_R , an iterative process must be used whereby the initially assumed moments are successively revised. However, the study reported in Ref. 31 seems to indicate that the first cycle will often be sufficiently accurate for most engineering applications.

6. From the stress-strain and strain-displacement relationships, the membrane loads N_θ and N_ϕ due to v and w can now be found.

7. The final approximate values for the bending moments, membrane loads, and displacements are found as follows:

a. Final $M_\theta = M_R$ + corrective M_θ .

b. Final $M_\phi = M_R$ + corrective M_ϕ .

c. Final N_θ and N_ϕ = obtained from step 6.

d. Final v and w = obtained from step 4.

(129)

The total approximate values for the stresses are obtained by superimposing those associated with the final bending moments, final membrane loads, and any self-equilibrating temperature distributions through the wall thickness.

To focus attention on the general concepts proposed by Christensen [33], no mention is made in the foregoing steps of the need to satisfy prescribed boundary conditions at locations removed from the apex. Therefore, it might now be helpful to note that, for this method, it is probably best first to obtain results under the assumption that no external constraints are present at such positions. Following this, edge forces and/or moments may be superimposed which enforce the required conditions at the boundaries.

The foregoing approaches are only two of a number of possibilities for the subject problem and can be used to obtain approximate values without the need for sophisticated mathematical and/or numerical operations. However, solutions can also be obtained by the use of existing digital computer programs, many of which use either discrete-element or finite-difference methods. Such programs are probably the best approach for obtaining rapid, accurate solutions. However, to retain a physical feel for the problem, it would be helpful to convert temperature distributions into equivalent mechanical loadings, such as was done for isotropic circular cylinders. It is recommended that future efforts include work along these lines to arrive at the equivalent pressures for shells of revolution having arbitrary shapes.

I. Sphere Under Radial Temperature Variation.

A. Hollow Sphere.

Inside radius = a .

Outside radius = b .

$$\sigma_{rr} = \frac{2\alpha E}{1 - \nu}\left[\frac{r^3 - a^3}{(b^3 - a^3)r^3}\int_a^b Tr^2\,dr - \frac{1}{r^3}\int_a^r Tr^2\,dr\right] ,$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{\alpha E}{1 - \nu}\left[\frac{a^3 + 2r^3}{(b^3 - a^3)r^3}\int_a^b Tr^2\,dr + \frac{1}{r^3}\int_a^{r_1} Tr^2\,dr - T\right] ,$$

$$u = \alpha\left(\frac{1 + \nu}{1 - \nu}\right)\frac{1}{b^3 - a^3}\left[\frac{a^3}{r^2}\int_r^b Tr^2\,dr + \frac{b^3}{r^2}\int_a^r Tr^2\,dr + \frac{2(1 - 2\nu)r}{(1 + \nu)}\int_a^b Tr^2\,dr\right] ,$$

$$T(r) = t_0 = \text{constant},$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0 ,$$

and

$$u = \alpha T_0 r .$$

B. **Solid Sphere.**

$$\sigma_{rr} = \frac{2\alpha E}{1-\nu}\left(\frac{1}{b^3}\int_0^b Tr^2\,dr - \frac{1}{r^3}\int_0^r Tr^2\,dr\right) ,$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^3}\int_0^b Tr^2\,dr + \frac{1}{r^3}\int_0^r Tr^2\,dr - T\right) ,$$

$$u = \alpha\left(\frac{1+\nu}{1-\nu}\right)\left[\frac{1}{r^2}\int_a^r Tr^2\,dr + \frac{2(1-\nu)}{(1+\nu)}\left(\frac{r}{b^3}\right)\int_0^b Tr^2\,dr\right] ,$$

$$\sigma_{rr}(\theta) = \sigma_{\theta\theta}(0) = \sigma_{\phi\phi}(0) = \frac{2\alpha E}{1-\nu}\left[\frac{1}{b^3}\int_0^b Tr^2\,dr - \frac{T(0)}{3}\right] ,$$

$$T(r) = T_0 = \text{constant} ,$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0 ,$$

and

$$u = \alpha T_0 r .$$