3.0.6.2 Statically Indeterminate.

The forces in the bars of a statically indeterminate pin-jointed truss are not zero, but are easily determined from the results of the previous paragraph.

3.0.7 Plates.

The analysis of plates presented in this section is based on the following classical theory of plates assumptions:

1. The material is isotropic, homogeneous, and linearly elastic.

2. The constant thickness of the plate is small when compared with its other dimensions.

3. Plane sections, which before bending are normal to the median plane of the plate, remain plane and normal to the median plane after bending.

4. The deflections of the plate are of the order of magnitude of the plate thickness.

Solutions are given herein for circular and rectangular plates with various boundary conditions and temperature distributions.

Analysis techniques for plates made of composite materials are given in Section F.

3.0.7.1 Circular Plates.

I. Temperature Gradient Through the Thickness.

A. Configuration.

The design curves and equations presented here apply only to flat, circular plates having central circular holes (Fig. 3.0-14). The plates must be of constant thickness, be made of an isotropic material, and obey Hooke's law. Curves are given which deal with bending phenomena for a/b ratios of 0.2, 0.4, 0.6, and 0.8 and are based on the assumption that $\nu = 0.30$. These plots cover only a portion of the boundary conditions considered in connection with bending behavior. The remainder of these conditions, as well as all the membrane solutions, are given as closed-form algebraic equations which are valid for arbitrary values of Poisson's ratio and the inner and outer radii of the plate. When the ratio a/b approaches unity, the member is more properly identified as a ring.

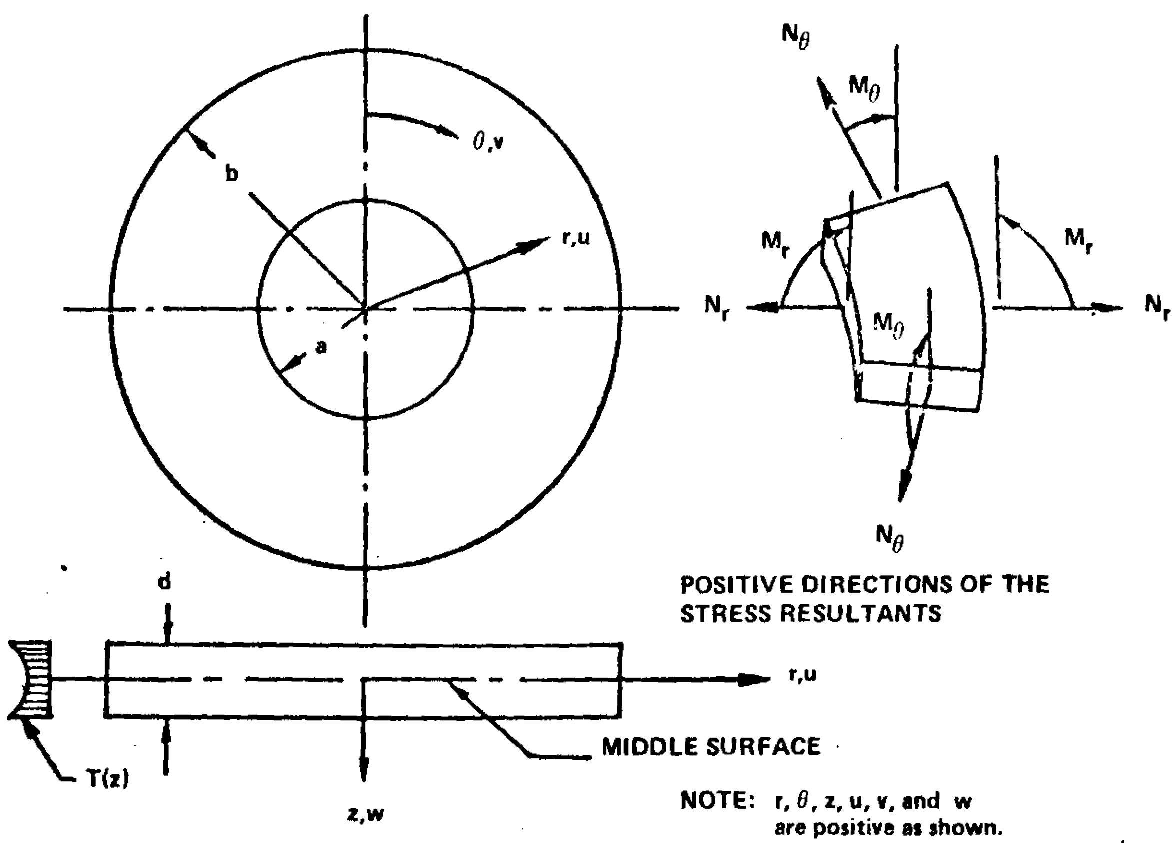

Figure 3.0-14. Circular flat plate.

B. Boundary Conditions.

Solutions are given for various combinations of the following boundary conditions for bending and membrane behavior, respectively.

Bending Phenomena.

1. Clamped, that is,

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = b \quad \text{and/or} \quad r = a \; ;$$

2. Simply supported, that is,

$$w = M_r = 0 \quad \text{at} \quad r = b \quad \text{and/or} \quad r = a \; ;$$

3. Free, that is,

$$\left.\begin{aligned} &M_r = 0 \\ &\frac{\partial M_r}{\partial r} - 2\frac{\partial M_{r\theta}}{\partial \theta} = 0 \end{aligned}\right\} \quad \text{at} \quad r = b \quad \text{and/or} \quad r = a \; .$$

Membrane Phenomena.

1. Radially fixed, that is,

$$\overline{u} = \overline{v} = 0 \quad \text{at} \quad r = b \quad \text{and/or} \quad r = a \; ;$$

2. Radially free, that is,

$$N_r = N_{r\theta} = 0 \quad \text{at} \quad r = b \quad \text{and/or} \quad r = a \; .$$

C. Temperature Distribution.

Arbitrary temperature variations may be present through the plate thickness. However, it is required that there be no gradients over the surface. Hence, the permissible distributions can be expressed in the form

$$T = T(z) \quad .$$

D. Design Curves and Equations.

In Ref. 2, Newman and Forray present the simple method given here to determine the thermal stresses and deflections for stable plates that satisfy the foregoing requirements. This technique is based on classical small-deflection theory and is an extension of the procedures published in Refs. 3 through 5 by the same authors. To perform the analysis, use is made of a number of equations and design curves. These are provided in the summary which follows.

It is assumed that Young's modulus and Poisson's ratio are unaffected by temperature variations. On the other hand, the temperature-dependence of the thermal-expansion coefficient can be accounted for by recognizing that it is the product αT which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α.

E. Summary of Equations and Curves.

$$w = -\frac{w' a^2 M_T}{D_b(1 - \nu^2)} \quad ,$$

$$\sigma_r = \frac{1}{t}\left[N_r + \frac{N_T}{(1-\nu)}\right] + \frac{12z}{t^3}\left[M_r + \frac{M_T}{(1-\nu)}\right] - \frac{E\alpha T}{(1-\nu)} \quad ,$$

and

$$\sigma_\theta = \frac{1}{t}\left[N_\theta + \frac{N_T}{(1-\nu)}\right] + \frac{12z}{t^3}\left[M_\theta + \frac{M_T}{(1-\nu)}\right] - \frac{E\alpha T}{(1-\nu)}$$

where

$$D_b = \frac{Et^3}{12\,(1-\nu^2)}$$

$$M_T = E\alpha \int_{-t/2}^{t/2} Tz\,dz \quad ,$$

$$N_T = E\alpha \int_{-t/2}^{t/2} T\,dz \quad ,$$

$$M_r = -12\, M_T M_r' \quad ,$$

$$M_\theta = -12\, M_T M_\theta' \quad ,$$

$$N_r = N_T N_r' \quad ,$$

and

$$N_\theta = N_T N_\theta' \quad .$$

The values for w', M_r', and M_θ' are obtained either from Table 3.0-5 or Figures 3.0-15 through 3.0-19. When the figures are used, Poisson's ratio must be taken to be 0.3 throughout the analysis of bending phenomena. When Table 3.0-5 is employed, there are no restrictions on ν. Also note that, in most of the plots, the parameters include multiplication factors. The values for N_r' and N' are obtained from Table 3.0-6 and are also valid for any value of Poisson's ratio.

F. Linear Gradient.

For the special case of a linear temperature gradient through the thickness for a solid plate represented by

$$T(z) = \frac{T_0 + T_1}{2} + \frac{T_0 - T_1}{2} z$$

the following solutions apply.

Unrestrained Solid Circular Plate.

$$\sigma_r = \sigma_\theta = 0 \quad .$$

The plate becomes curved and fits a sphere of radius inversely proportional to the difference in surface temperatures and directly proportional to the thickness.

Clamped Plate.

$$w = 0$$

TABLE 3.0-5. NONDIMENSIONAL PARAMETERS FOR BENDING PHENOMENA

Boundary Conditions				
Outer Edge	Inner Edge	w'	M_r'	M_θ'
Free	Free	$\frac{1}{2}\left(\frac{r^2}{a^2} - 1\right)$ (Relative to Inner Boundary)	0	0
Clamped	Clamped	0	$\frac{1}{12(1-\nu)}$	$\frac{1}{12(1-\nu)}$
Simply Supported	Free	$\frac{1}{2}\left(\frac{r^2}{a^2} - \frac{b^2}{a^2}\right)$	0	0
Free	Simply Supported	$\frac{1}{2}\left(\frac{r^2}{a^2} - 1\right)$	0	0

TABLE 3.0-6. NONDIMENSIONAL PARAMETERS FOR MEMBRANE PHENOMENA

Boundary Conditions			
Outer Edge	Inner Edge	N_r'	N_θ'
Radially Free	Radially Free	0	0
Radially Fixed	Radially Free	$\frac{\left[\left(\frac{a}{r}\right)^2 - 1\right]}{\left[(1+\nu)\left(\frac{a}{b}\right)^2 + (1-\nu)\right]}$	$\frac{-\left[\left(\frac{a}{r}\right)^2 + 1\right]}{\left[(1+\nu)\left(\frac{a}{b}\right)^2 + (1-\nu)\right]}$
Radially Free	Radially Fixed	$\frac{\left[\left(\frac{b}{r}\right)^2 - 1\right]}{\left[(1+\nu)\left(\frac{b}{a}\right)^2 + (1-\nu)\right]}$	$\frac{-\left[\left(\frac{a}{r}\right)^2 + 1\right]}{\left[(1+\nu)\left(\frac{b}{a}\right)^2 + (1-\nu)\right]}$
Radially Fixed	Radially Fixed	$-\frac{1}{(1-\nu)}$	$-\frac{1}{(1-\nu)}$

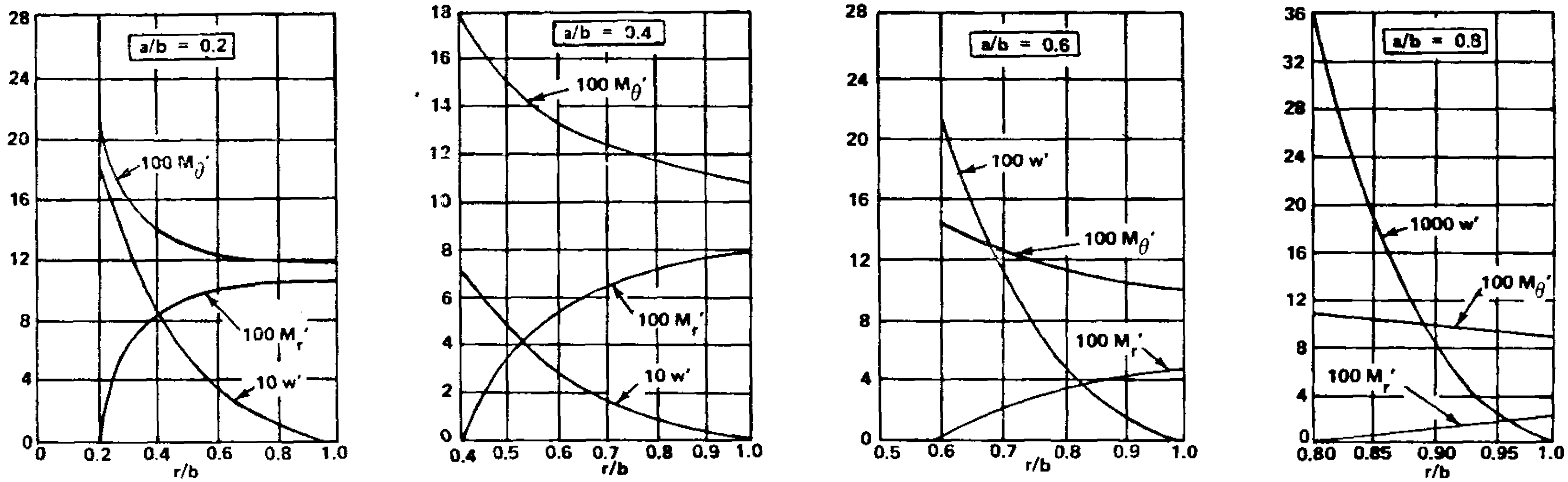

Figure 3.0-15. Nondimensional parameters for bending phenomena; outer edge clamped and inner edge free (ν=0.3).

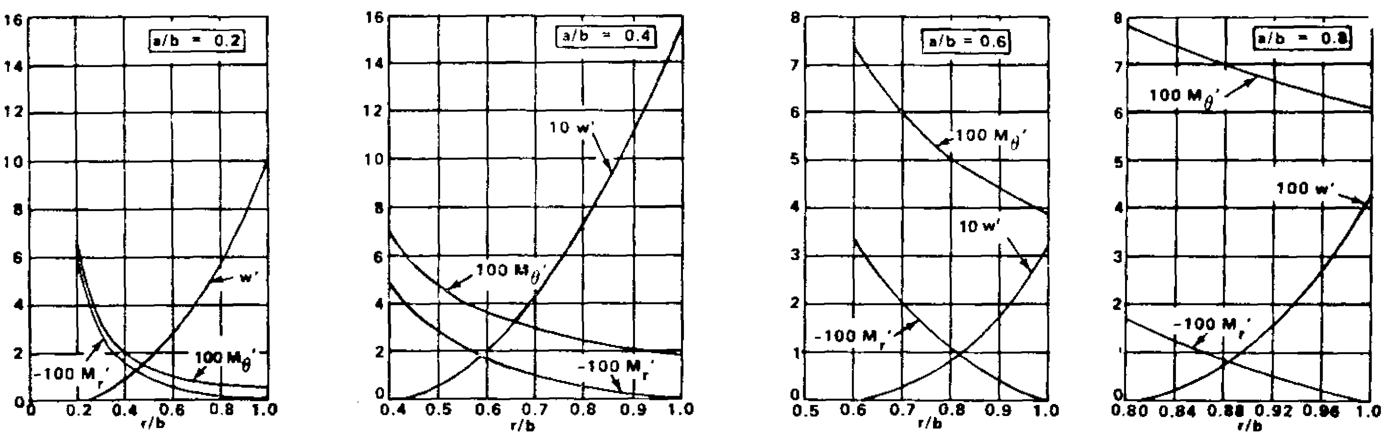

Figure 3.0-16. Nondimensional parameters for bending phenomena; outer edge free and inner edge clamped (ν=0.3).

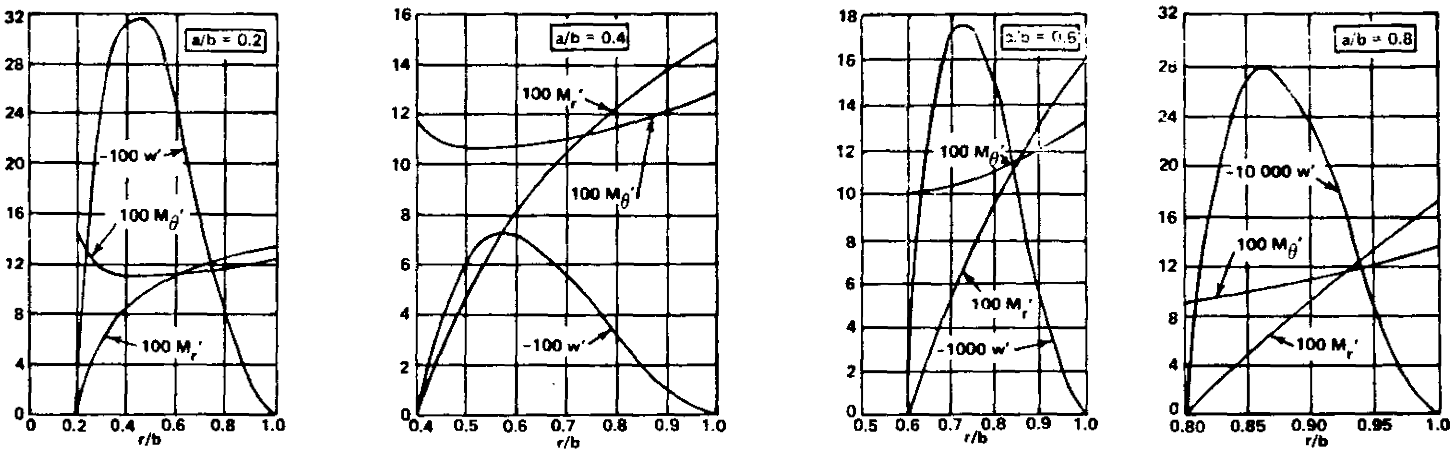

Figure 3.0-17. Nondimensional parameters for bending phenomena; outer edge clamped and inner edge simply supported (ν=0.3).

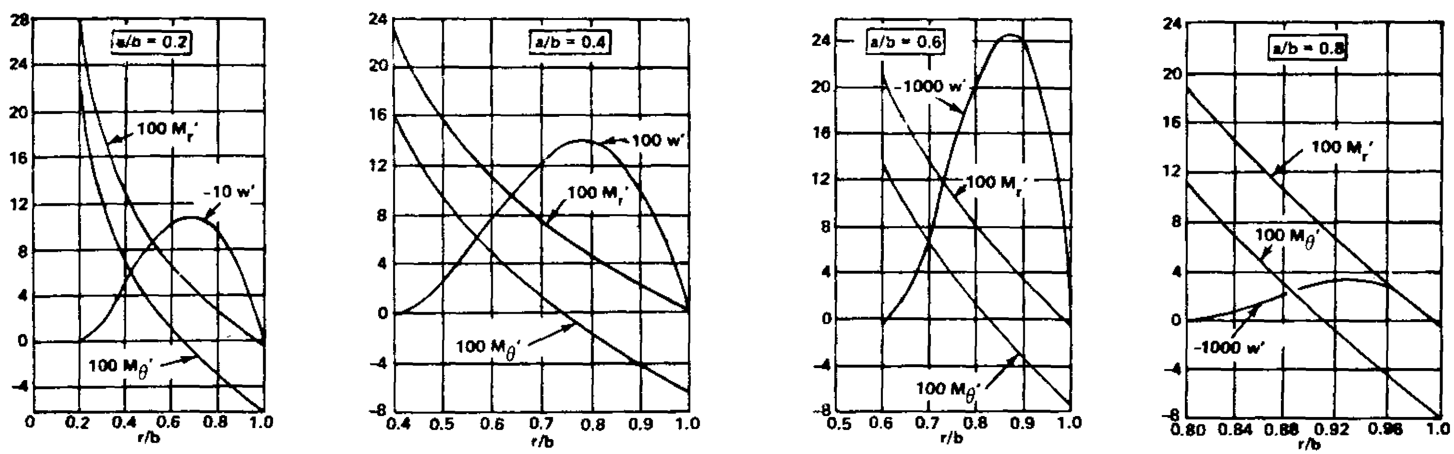

Figure 3.0-18. Nondimensional parameters for bending phenomena; outer edge simply supported and inner edge clamped (ν=0.3).

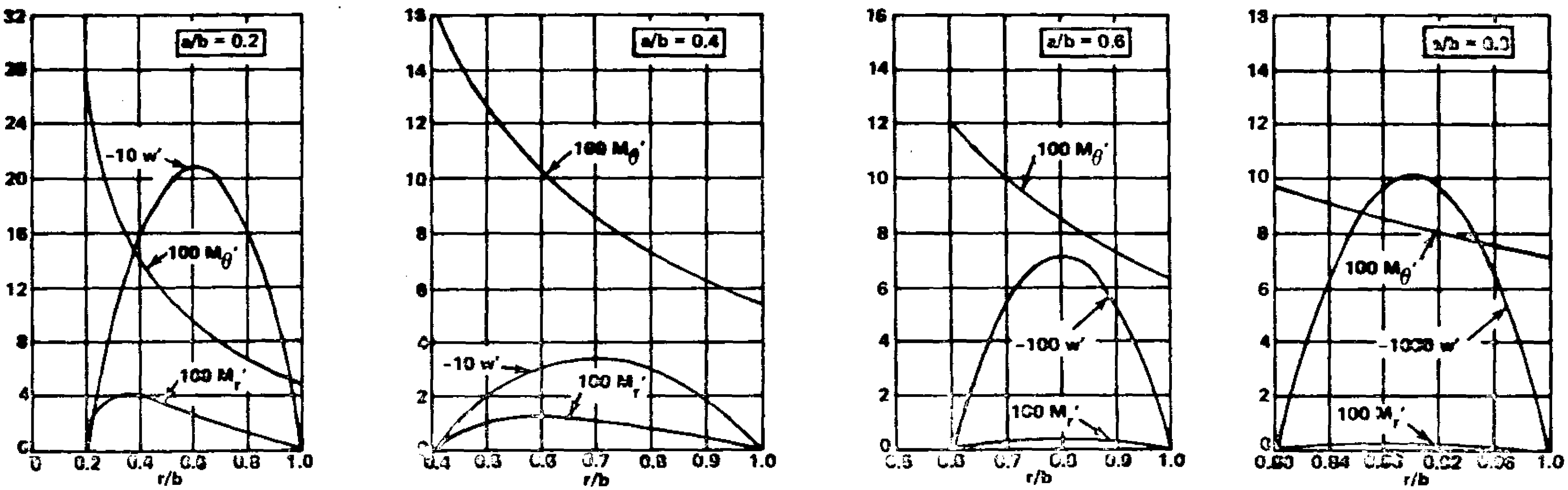

Figure 3.0-19. Nondimensional parameters for bending phenomena; outer edge simply supported and inner edge simply supported (ν=0.3).

and

$$\sigma_r|_{max} = \sigma_\theta|_{max} = \frac{\alpha E(T_1 - T_0)}{2(1-\nu)} \quad .$$

Simply Supported Plate.

$$\sigma_r = \sigma_\theta = 0$$

and

$$w = \frac{-\alpha(T_1 - T_0)}{2t}(a^2 - r^2) \quad .$$

II. Temperature Difference as a Function of the Radial Coordinates.

A. Clamped Plate.

For the variation of temperature, assumed to be linear through the thickness, and the variation with r given by the monomial,

$$T = \frac{z}{h} A_\kappa r^\kappa + c$$

where A_κ and c are constants.

$$T_D = A_\kappa r^\kappa \quad ,$$

$$T_D(a) = A_\kappa a^\kappa \quad ,$$

$$w = \frac{(1+\nu)\,a^2\alpha T_D(a)}{(\kappa+2)^2}\left[\left(\frac{r}{a}\right)^{\kappa+2} - \left(\frac{\kappa}{2}+1\right)\left(\frac{r}{a}\right)^2 + \frac{\kappa}{2}\right] ,$$

$$M_r = \frac{Eh^2\alpha T_D(a)}{12(\kappa+2)}\left[\left(\frac{r}{a}\right)^{\kappa} + \frac{1+\nu}{1-\nu}\right]$$

and

$$M_\theta = \frac{Eh^2\alpha T_D(a)}{12(\kappa+2)}\left[(\kappa+1)\left(\frac{r}{a}\right)^{\kappa} + \frac{1+\nu}{1-\nu}\right] ,$$

where T_D is the temperature difference between the surfaces.

Curves of nondimensional deflection and moments are presented in Figures 3.0-20 through 3.0-22 for $\kappa = 1, 2, \ldots 5$. Superposition may then be used for T_D given by polynomials in r . The determination of a polynomial describing the radial variation of T_D can be obtained in the same manner as shown in Paragraph 3.0.2.3.

B. Radially Fixed or Radially Free Plate.

Configuration.

The design equations provided here apply only to flat, circular plates which may or may not have a central circular hole. The plate must be of constant thickness, be made of an isotropic material, and obey Hooke's law. Formulas are given which cover the range

$$0 \leq \frac{a}{b} < 1 \quad .$$

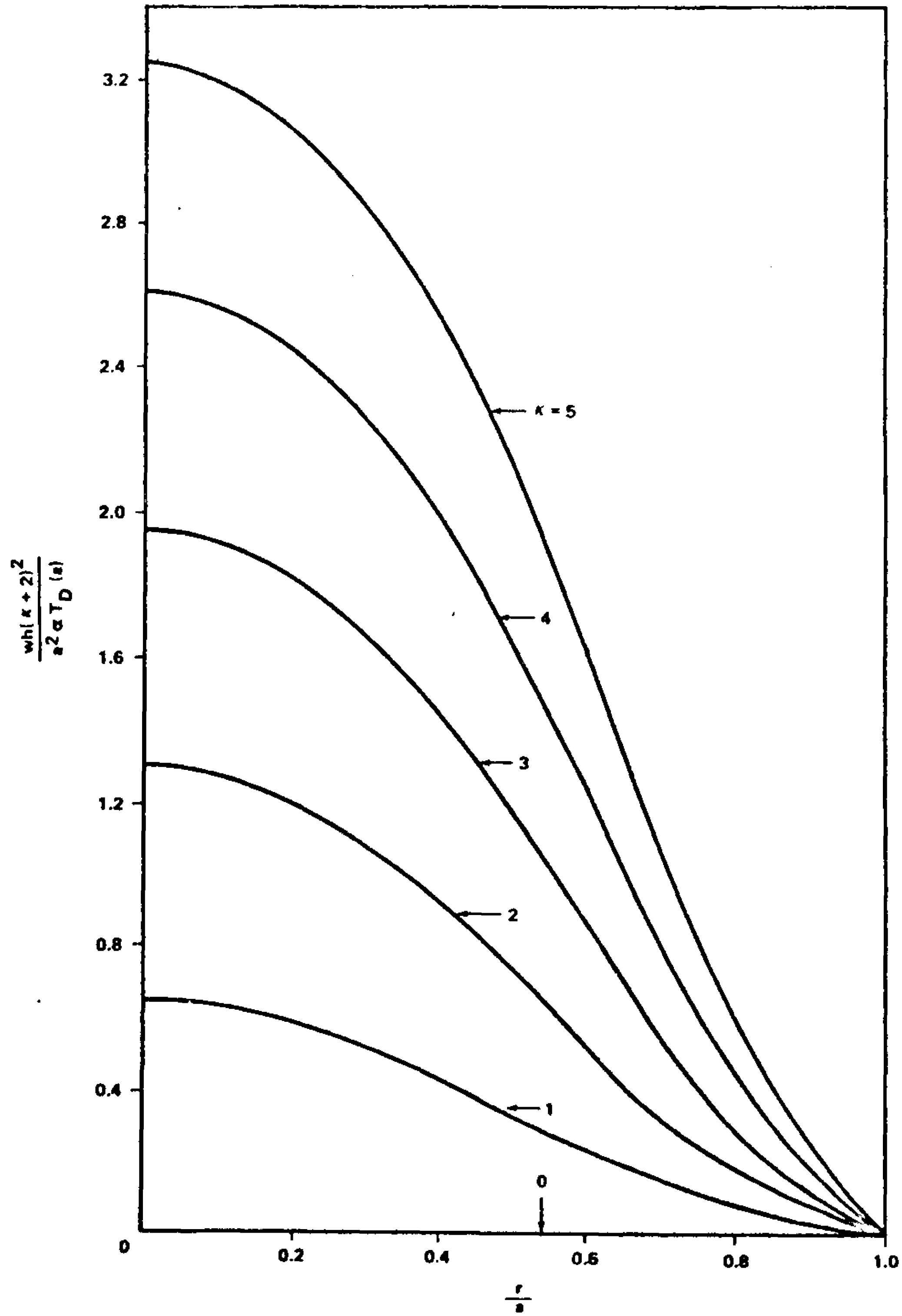

Figure 3.0-20. Nondimensional deflection.

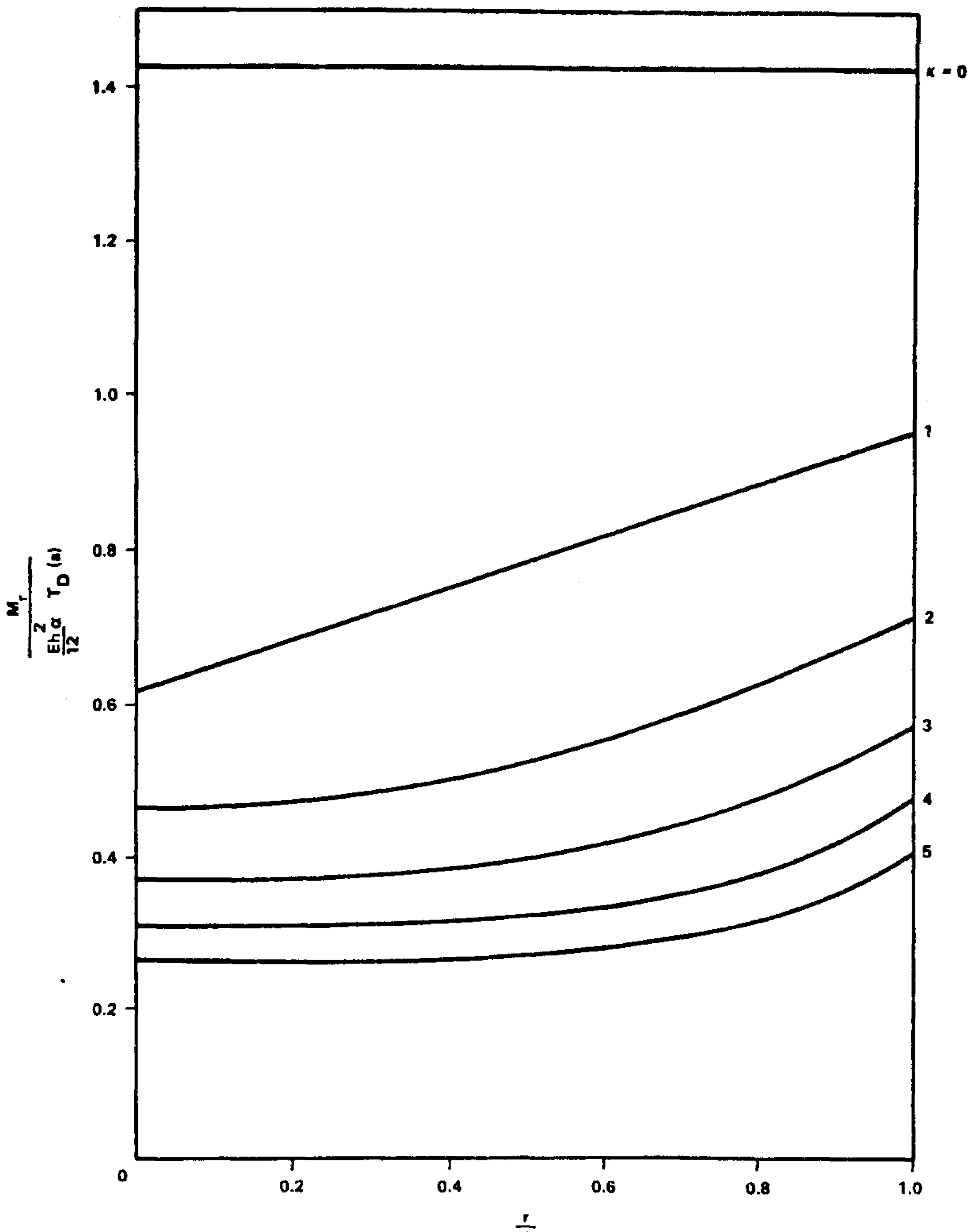

Figure 3.0-21. Nondimensional radial moment.

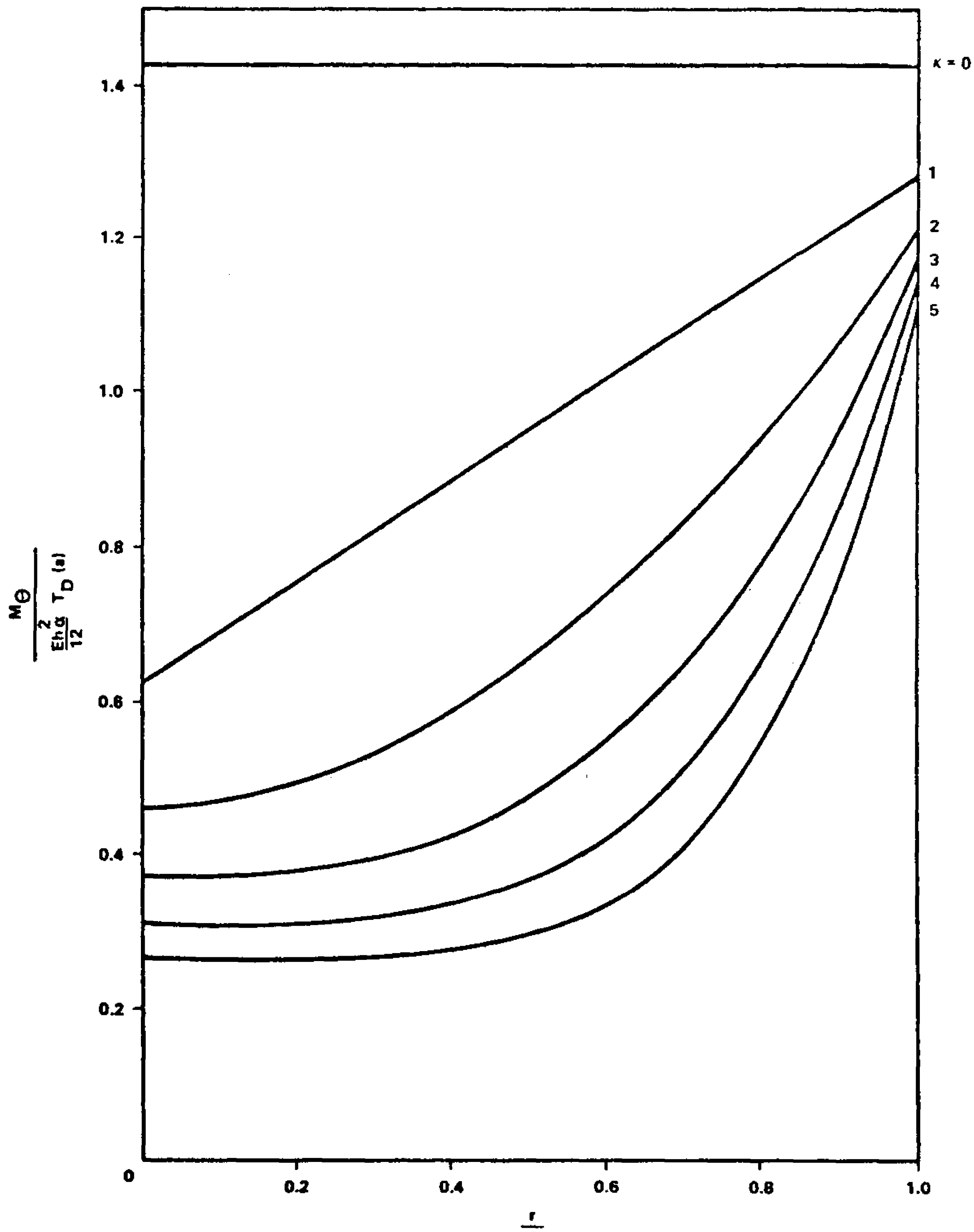

Figure 3.0-22. Nondimensional tangential moment.

As this ratio approaches unity, the member is more properly identified as a ring.

Boundary Conditions.

Solutions are given for each of the following types of boundary conditions.

1. Radially free; that is,

$$\sigma_r = \sigma_{r\theta} = 0 \quad \text{at} \quad r = b$$

and

$$\sigma_r = \sigma_{r\theta} = 0 \quad \text{at} \quad r = a$$

if hole is present.

2. Outer boundary radially fixed (if hole is present, the inner boundary is radially free); that is,

$$u = v = 0 \quad \text{at} \quad r = b$$

and

$$\sigma_r = \sigma_{r\theta} = 0 \quad \text{at} \quad r = a$$

if hole is present.

Temperature Distribution.

The supposition is made that the temperature is uniform through the thickness. However, the plate may be subjected to a surface distribution which has an arbitrary radial gradient but no circumferential variations. Hence, the permissible distributions can be expressed in the form

$$T = T(r) \quad .$$

Design Equations.

In this section, it is assumed that Young's modulus and Poisson's ratio are unaffected by temperature changes. Therefore, the user must select single effective values for each of these properties by using some type of averaging technique. The same approach may be taken with regard to the coefficient of thermal expansion. On the other hand, the temperature-dependence of this property can be accounted for by recognizing that it is the product αT which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α.

The design equations are given in the summary which follows and are based on classical small-deflection theory. The expressions for the radially free plates were taken directly from Ref. 1. The equations for plates with radially fixed outer boundaries were derived by superposition of the free-plate formulas and the relationships given in Ref. 6 for cylinders subjected solely to external radial pressure. Depending upon the complexity of the temperature distribution, the required integrations may be performed either analytically or by numerical procedures.

Summary of Equations.

Solid Plates (No Central Hole).

1. Radially free boundary:

$$\sigma_r = \alpha E \left(\frac{1}{b^2} \int_o^b Tr\, dr - \frac{1}{r^2} \int_o^r Tr\, dr \right) ,$$

$$\sigma_\theta = \alpha E \left(-T + \frac{1}{b^2} \int_o^b Tr\, dr + \frac{1}{r^2} \int_o^r Tr\, dr \right) ,$$

and

$$u = \frac{\alpha}{r} \left[(1+\nu) \int_o^r Tr\, dr + (1-\nu)\left(\frac{r}{b}\right)^2 \int_o^b Tr\, dr \right] .$$

These three equations are indeterminate at $r = 0$. However, by the application of l'Hospital's rule, it is found that [1]

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = \alpha E \left[\frac{1}{b^2} \int_o^b Tr\, dr - \frac{1}{2}(T)_{r=0} \right]$$

and

$$u_{r=0} = 0 .$$

2. Radially fixed boundary:

$$\sigma_r = \alpha E \left[\frac{1}{b^2} \int_o^b Tr\,dr - \frac{1}{r^2} \int_o^r Tr\,dr - \frac{2}{b^2(1-\nu)} \int_o^b Tr\,dr\right] ,$$

$$\sigma_\theta = \alpha E \left[-T + \frac{1}{b^2} \int_o^b Tr\,dr + \frac{1}{r^2} \int_o^r Tr\,dr - \frac{2}{b^2(1-\nu)} \int_o^b Tr\,dr\right] ,$$

and

$$u = \frac{\alpha}{r}\left[(1+\nu) \int_o^r Tr\,dr - (1+\nu)\left(\frac{r}{b}\right)^2 \int_o^b Tr\,dr\right] .$$

These three equations are indeterminate at $r = 0$. However, by the application of l'Hospital's rule, it is found that

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = \alpha E \left[\frac{1}{b^2} \int_o^b Tr\,dr - \frac{2}{b^2(1-\nu)} \int_o^b Tr\,dr - \frac{1}{2} T_{r=0}\right]$$

and

$$u_{r=0} = 0 .$$

Plates with Central Hole.

1. Both boundaries radially free:

$$\sigma_r = \frac{\alpha E}{r^2}\left(\frac{r^2 - a^2}{b^2 - a^2} \int_a^b Tr\,dr - \int_a^r Tr\,dr\right) ,$$

$$\sigma_\theta = \frac{\alpha E}{r^2}\left(\frac{r^2 + a^2}{b^2 - a^2}\int_a^b Tr\,dr + \int_a^r Tr\,dr - Tr^2\right) ,$$

and

$$u = \frac{\alpha}{r}\left[(1+\nu)\int_a^r Tr\,dr + \frac{(1-\nu)r^2 + (1+\nu)a^2}{b^2 - a^2}\int_a^b Tr\,dr\right] .$$

2. Inner boundary radially free and outer boundary radially fixed:

$$\sigma_r = \frac{\alpha E}{r^2}\left[\frac{r^2 - a^2}{b^2 - a^2}\int_a^b Tr\,dr - \int_a^r Tr\,dr\right] + (\sigma_r)_{r=b}\left[\frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)\right] ,$$

$$\sigma_\theta = \frac{\alpha E}{r^2}\left[\frac{r^2 + a^2}{b^2 - a^2}\int_a^b Tr\,dr + \int_a^r Tr\,dr - Tr^2\right]$$

$$+ (\sigma_r)_{r=b}\left[\frac{b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)\right] ,$$

$$u = \frac{\alpha}{r}\left[(1+\nu)\int_a^r Tr\,dr + \frac{(1-\nu)r^2 + (1+\nu)a^2}{(b^2 - a^2)}\int_a^b Tr\,dr\right]$$

$$+ (\sigma_r)_{r=b}\left\{\frac{b^2}{E(b^2 - a^2)}\left[(1-\nu)r + \frac{(1+\nu)}{r}a^2\right]\right\} ,$$

where

$$(\sigma_r)_{r=b} = -\frac{\alpha E}{b^2}\frac{1}{\left(\frac{b^2 + a^2}{b^2 - a^2} - \nu\right)}\left[(1+\nu)\int_a^b Tr\,dr\right.$$

$$+ \frac{(1-\nu)b^2 + (1+\nu)a^2}{(b^2 - a^2)} \int_a^b Tr\,dr \Bigg] \quad .$$

III. Disk With Central Shaft.

Boundary conditions for this plane-stress problem are $u\Big|_{r=a} = 0$ and $\sigma_{rr}\Big|_{r=a} = 0$.

$$T = T_b \left(\frac{r-a}{b-a}\right)^n \quad ,$$

$$\frac{\sigma_{rr}}{\alpha E T_b} = \frac{1}{2}\left[\frac{T_b^*}{T_b}\left(1 - \frac{a^2}{b^2}\right)\frac{\left(\frac{1+\nu}{1-\nu} + \frac{a^2}{r^2}\right)}{\left(\frac{1+\nu}{1-\nu} + \frac{a^2}{r^2}\right)} - \frac{T^*}{T_b}\left(1 - \frac{a^2}{r^2}\right)\right] \quad ,$$

$$\frac{\sigma_{\theta\theta}}{\alpha E T_b} = \frac{1}{2}\left[\frac{T_b^*}{T_b}\left(1 - \frac{a^2}{r^2}\right)\frac{\left(\frac{1+\nu}{1-\nu} + \frac{a^2}{r^2}\right)}{\left(\frac{1+\nu}{1-\nu} + \frac{a^2}{r^2}\right)} + \frac{T^*}{T_b}\left(1 - \frac{a^2}{r^2}\right) - \frac{2T}{T_b}\right] \quad ,$$

$$\frac{\sigma_{rr}}{\alpha E T_b}\Bigg|_{r=a} = \left\{\frac{T_b^*}{T_b}\left(1 - \frac{a^2}{b^2}\right)\left[\frac{1}{(1+\nu) + \frac{a^2}{b^2}(1-\nu)}\right]\right\} \quad ,$$

$$\frac{\sigma_{\theta\theta}}{\alpha E T_b}\Bigg|_{r=a} = \left\{\frac{T_b^*}{T_b}\left(1 - \frac{a^2}{b^2}\right)\left[\frac{\nu}{(1+\nu) + \frac{a^2}{b^2}(1-\nu)}\right]\right\} \quad ,$$

$$\frac{\sigma_{\theta\theta}}{\alpha E T_b}\Bigg|_{r=b} = \left\{\frac{T_b^*}{T_b}\left(1 - \frac{a^2}{b^2}\right)\left[\frac{1}{1 + \left(\frac{1-\nu}{1+\nu}\right)\frac{a^2}{b^2}}\right] - 1\right\} \quad ,$$

and

$$T^* = \frac{2}{r^2 - a^2}\int_a^r Tr\,dr = \frac{2T_b}{r+a}\left(\frac{r-a}{b-a}\right)^n \frac{[(n+1)\ r + a]}{(n+1)(n+2)}$$

where

$$T_b^* = \frac{2}{r^2 - a^2} \int_a^b Tr\,dr = \frac{2T_b}{a+b} \frac{[(n+1)\ b + a]}{(n+1)(n+2)} .$$

Curves showing the variations of $\left.\frac{\sigma_{rr}}{\alpha ET_b}\right|_{r=a}$ and $\left.\frac{\sigma_{\theta\theta}}{\alpha ET_b}\right|_{r=b}$ with n and a/b are given in Figures 3.0-23 through 3.0-25.

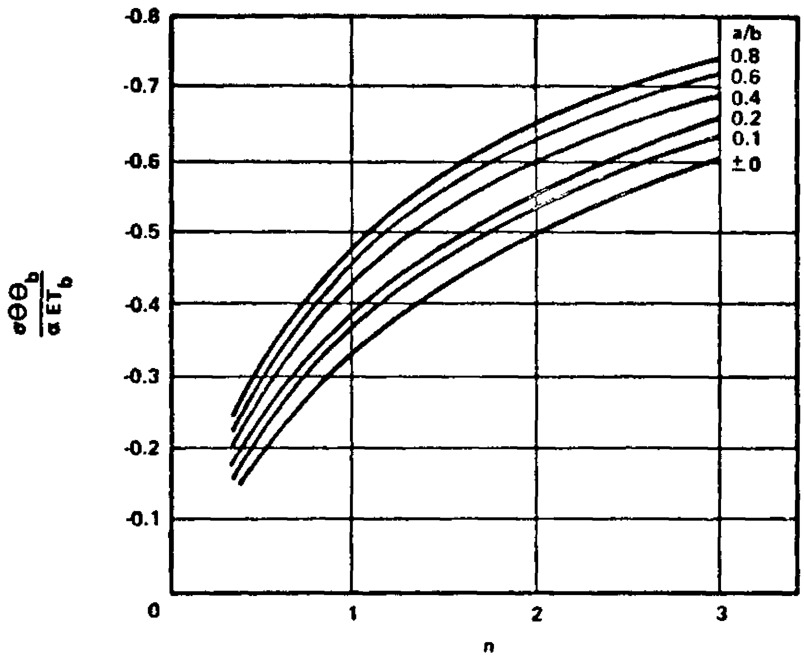

Figure 3.0-23. Variation of tangential stress at outer boundary with n and a/b for a disk on a shaft.

Additional cases that may be obtained from Refs. 7 and 8 are as follows:

1. Circular plate with asymmetrical temperature distribution

2. Circular disk with concentric hole subjected to asymmetrical temperature distribution

3. Circular plate with a central hot spot.

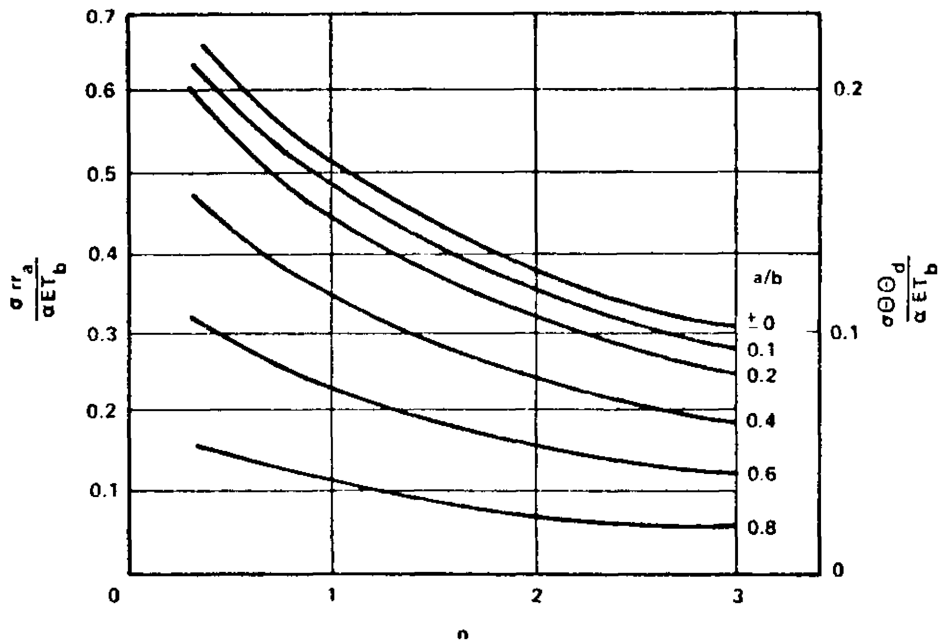

Figure 3.0-24. Variation of radial and tangential stress at inner boundary of disk with n and a/b for a disk on a shaft.

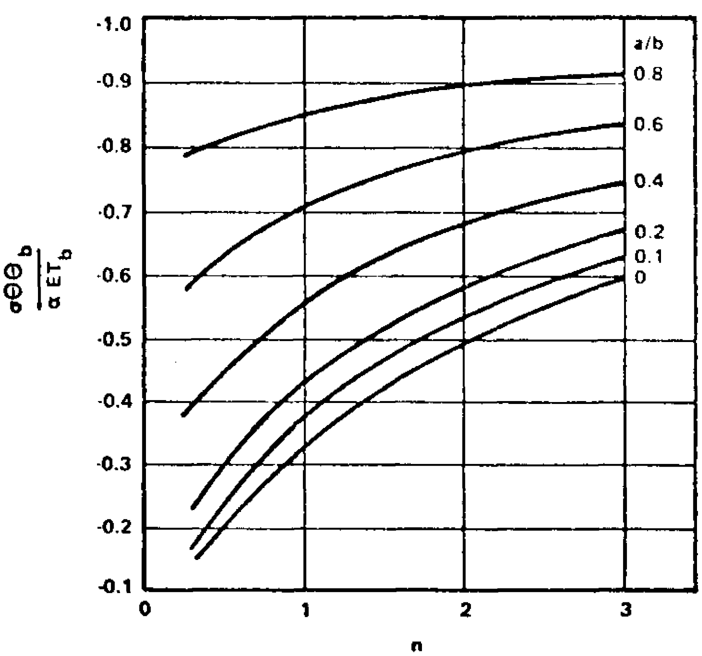

Figure 3.0-25. Variation of tangential stress at outer boundary of disk with n and a/b for a disk on a shaft.

3.0.7.2 Rectangular Plates.

I. Temperature Gradient Through the Thickness.

A. Configuration.

The design curves and equations provided here apply only to flat, rectangular plates which are of constant thickness and are made of isotropic material. The two long edges of the plate are supported by flexible beams. It is assumed that both the plate and the support beams are free of holes and that no stresses exceed the elastic limit in either of these members. The design curves cover aspect ratios b/a of 1.0, 1.5, and 3.0.

B. Boundary Conditions.

The edges $x = 0$ and $x = a$ are elastically supported by beams having equal flexural stiffnesses $E_b I_b$. Both beams are simply supported at their ends (Fig. 3.0-26) and are free to undergo axial expansions or contractions. These members offer no constraint to each of the following plate deformations:

1. Edge-Rotation
2. In-Place Edge-Displacements u and v.

The beams resist only transverse deflections w. The edges $y = 0$ and $y = b$ are simply supported; that is,

$$w = My = 0$$

along these two boundaries.

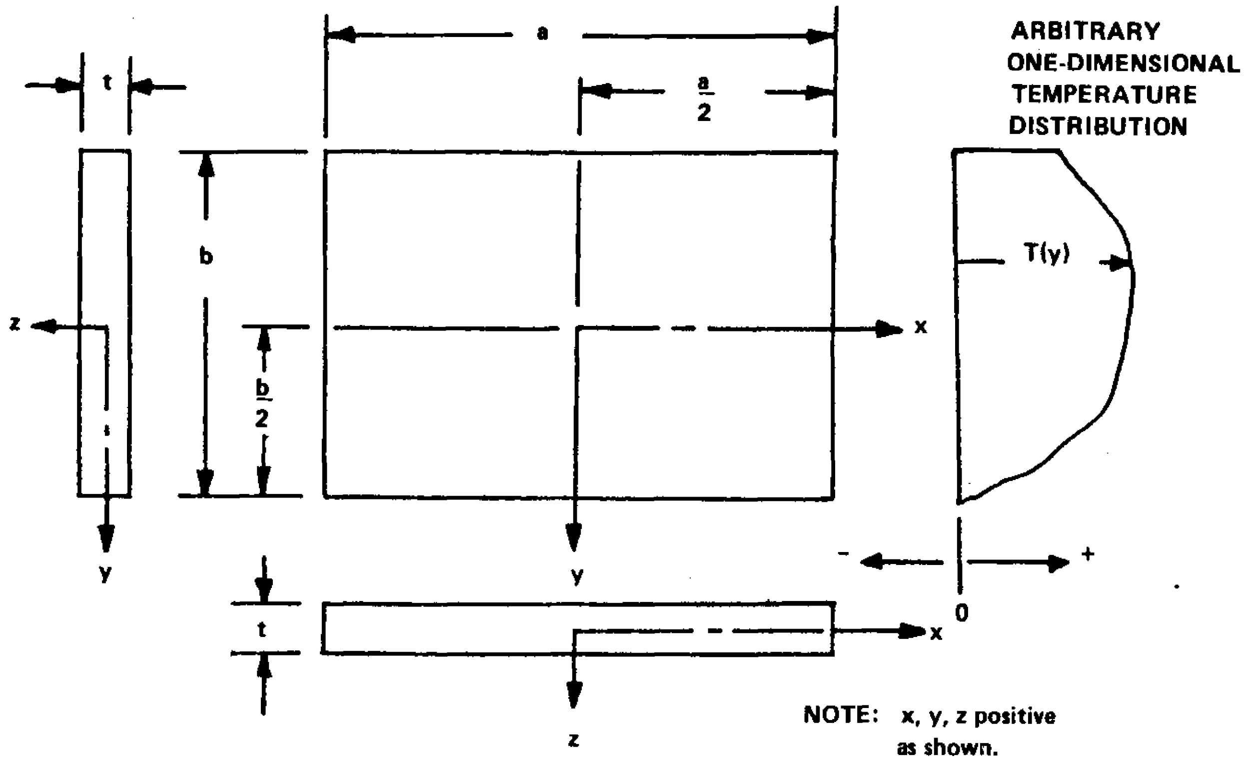

Figure 3.0-26. Rectangular flat plate with one-dimensional temperature distribution over surface.

C. Temperature Distribution.

Separate coverage is provided for each of the following temperature distributions through the thickness:

1. Linear gradient $T = a_0' + a_1'z$

2. Arbitrary gradient $T = f(z)$.

It is assumed that there are no temperature variations in directions parallel to the middle surface of the plate.

D. Design Curves and Equations.

In Ref. 9, Forray, et al., present the simple methods given here to compute thermal stresses and deflections at virtually any point in flat, rectangular plates which comply with the foregoing specifications. These techniques consist of a variety of equations and curves, all of which are based on the conventional small-deflection theory of plates. It is assumed that Young's modulus and Poisson's ratio are unaffected by temperature variations. Hence, the user must select single effective values for each of these properties by employing some type of averaging technique. The same approach may be taken with regard to the coefficient of thermal expansion. On the other hand, the temperature-dependence of this property can be accounted for by recognizing that it is the product T which governs; that is, the actual temperature distribution can be suitably modified to compensate for variations in α. When this approach is taken, the user must adopt the viewpoint that any mention of a linear temperature distribution is actually referring to a straight-line variation of the product αT.

Linear Temperature Gradient.

For a linear temperature gradient expressed by

$$T = a_0' + a_1' z$$

and with

$$T_D = T_{(z=t/2)} - T_{(z=t/2)} \quad ,$$

$$D_b = \frac{E_p t^3}{12(1 - \nu^2)}$$

and

$$M = \frac{4D_b \alpha T_D (1 - \nu^2)}{t} \quad .$$

The transverse deflections are expressed by

$$w = \overline{w}(x,y) + w^A(x,y) + w^A(a - x, y)$$

where

$$\overline{w}(x,y) = \frac{(1+\nu)\alpha T_D a^2}{2t} \left[\frac{x}{a}\left(1 - \frac{x}{a}\right) - \frac{8}{\pi^3} \sum_{n=1,3,\ldots}^{\infty}{}' \frac{\sin \frac{n\pi x}{a} \left[\cosh \frac{n\pi}{a}\left(y - \frac{b}{2}\right)\right]}{n^3 \cosh \frac{n\pi b}{2a}} \right] ,$$

$$w^A(x,y) = \sum_{m=1,3,\ldots}^{\infty} G_m^* \left[\frac{m\pi x}{b} \cosh \frac{m\pi x}{b} - \left(\frac{2}{1-\nu} + \frac{m\pi a}{b} \coth \frac{m\pi a}{b} \right) \sinh \frac{m\pi x}{b} \right] \sin \frac{m\pi y}{b}$$

and

$$G_m^* = \frac{- \frac{4M}{m\pi a} \left[\sum_{k=1,3,\ldots}^{2N-1}{}' \frac{1}{1 + \left(\frac{k}{m}\frac{b}{a}\right)^2} \right] \sinh \frac{m\pi a}{b}}{D_b \left(\frac{m\pi}{b}\right)^3 \left\{ \left(\cosh \frac{m\pi a}{b} - 1\right) \left[(3+\nu) \sinh \frac{m\pi a}{b} - (1-\nu) \frac{m\pi a}{b} \right] + \frac{2}{1-\nu} \left(\frac{E_b I_b}{D_b b} \right) m\pi \sinh^2 \left(\frac{m\pi a}{b}\right) \right\}}$$

The component $\overline{w}(x,y)$ is the deflection where all edges are simply supported.

The bending moments M_x and M_y can be obtained by substituting the final deflection relation into the following equations:

$$M_x = -D_b \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} + \frac{\alpha T_D}{t} (1 + \nu) \right]$$

and

$$M_y = -D_b \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} + \frac{\alpha T_D}{t} (1 + \nu) \right] \quad .$$

Forray et al. [9] used these deflection and moment expressions to generate the design curves of Figures 3.0-27 through 3.0-35, where it is assumed that $\nu = 0.30$. Some of the curves are discontinued near the corners of the plate $x/a = 0$, $y/b = 0$ since the conventional theory breaks down at these locations. Certain of these results also appear in Ref. 10, where a different plotting format was used. In addition, the latter reference includes supplementary curves for the case where $E_b I_b / D_b b \rightarrow \infty$. This corresponds to the condition of simple support on all four edges.

Because of symmetry about the centerlines $(x = a/2 \, , \; y = b/2)$, it was necessary to show only one quadrant of the plate in Figures 3.0-27 through 3.0-35. The assortment of curves covers a wide range of values in the variables and should accommodate most practical problems of this particular class. For any situations where the plots prove to be inadequate, the equations can be used to obtain solutions. However, a considerable amount of rather routine mathematics would be required in making the necessary substitutions to obtain bending-moment equations in series form. Once these were available, it might

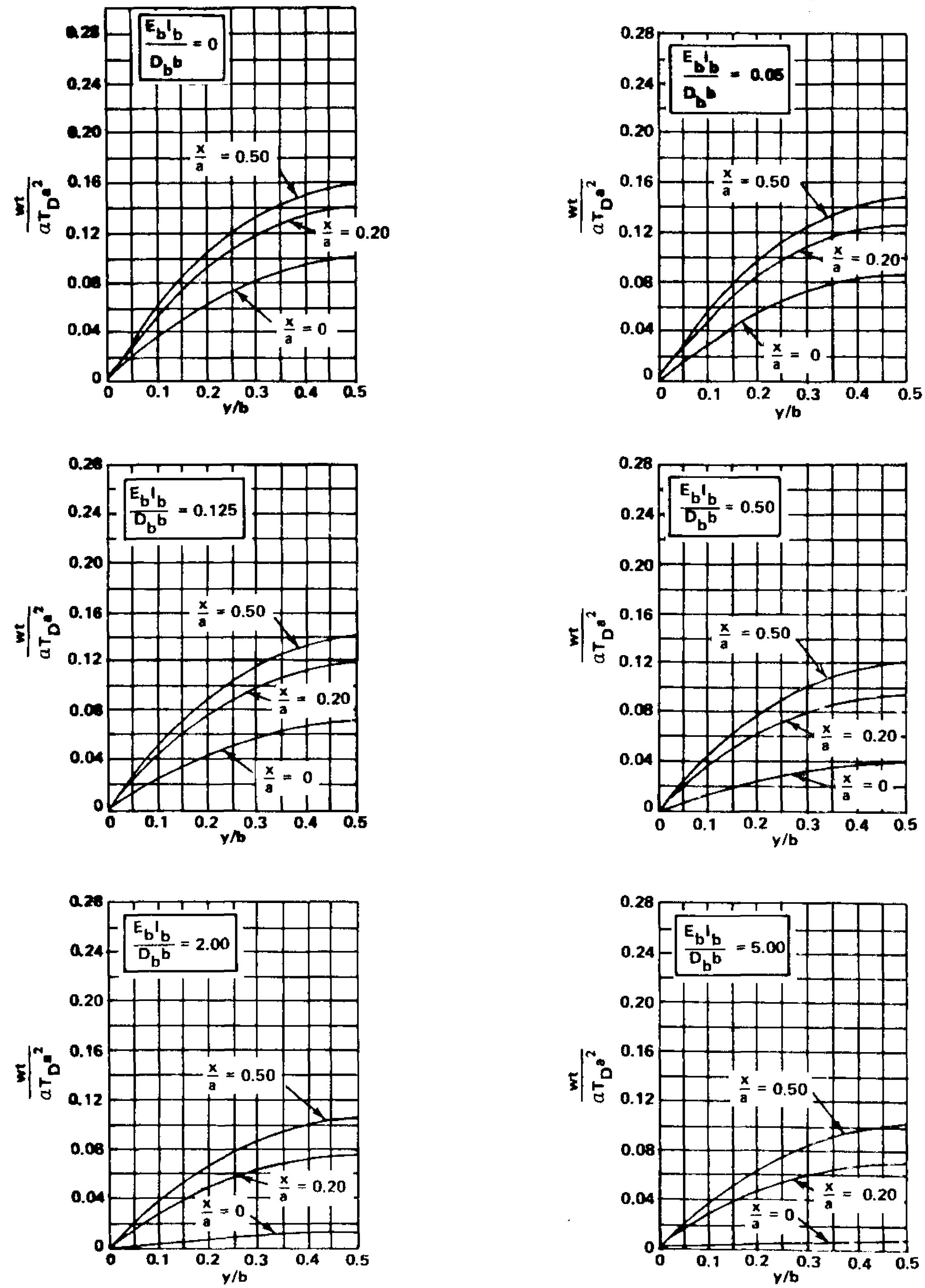

Figure 3.0-27. Nondimensional deflections for a plate with two opposite edges elastically supported and the other two edges simply supported; $b/a = 1.0$, $\nu = 0.30$.

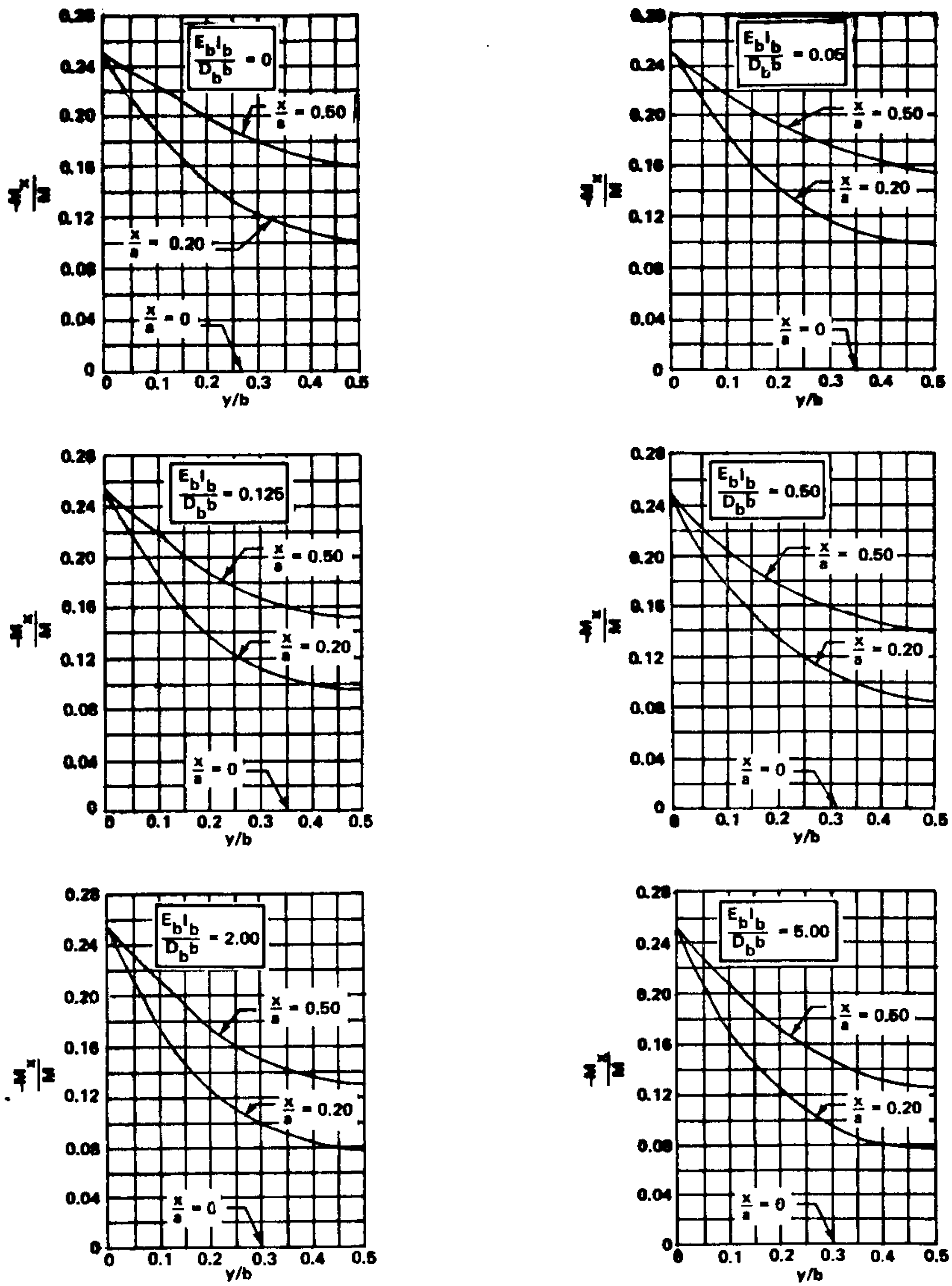

Figure 3.0-28. Nondimensional bending moments M_x/M for a plate with two opposite edges elastically supported and the other two edges simply supported; $b/a = 1.0$, $\nu = 0.30$.

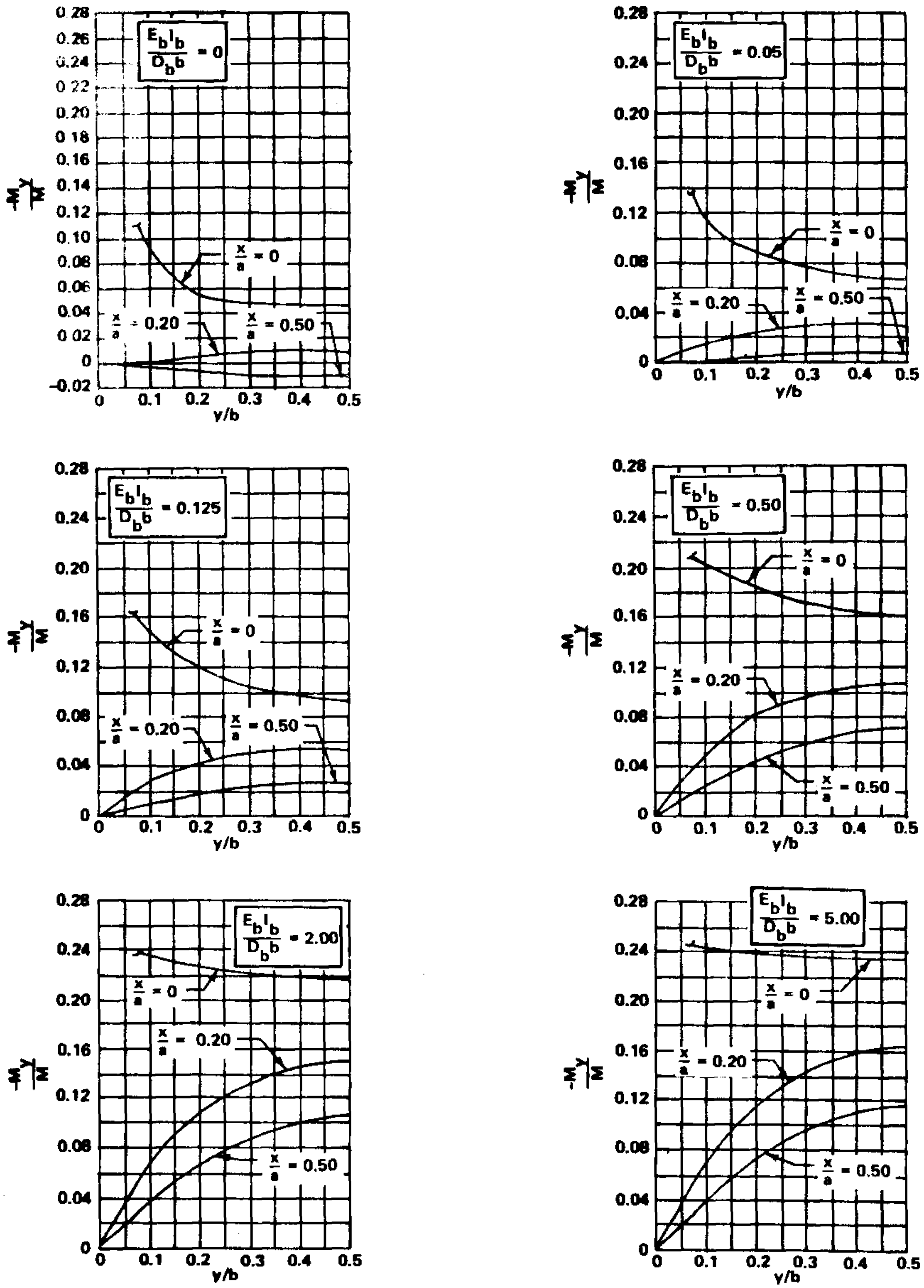

Figure 3.0-29. Nondimensional bending moments M_y/M for a plate with two opposite edges elastically supported and the other two edges simply supported; $b/a = 1.0$, $\nu = 0.30$.

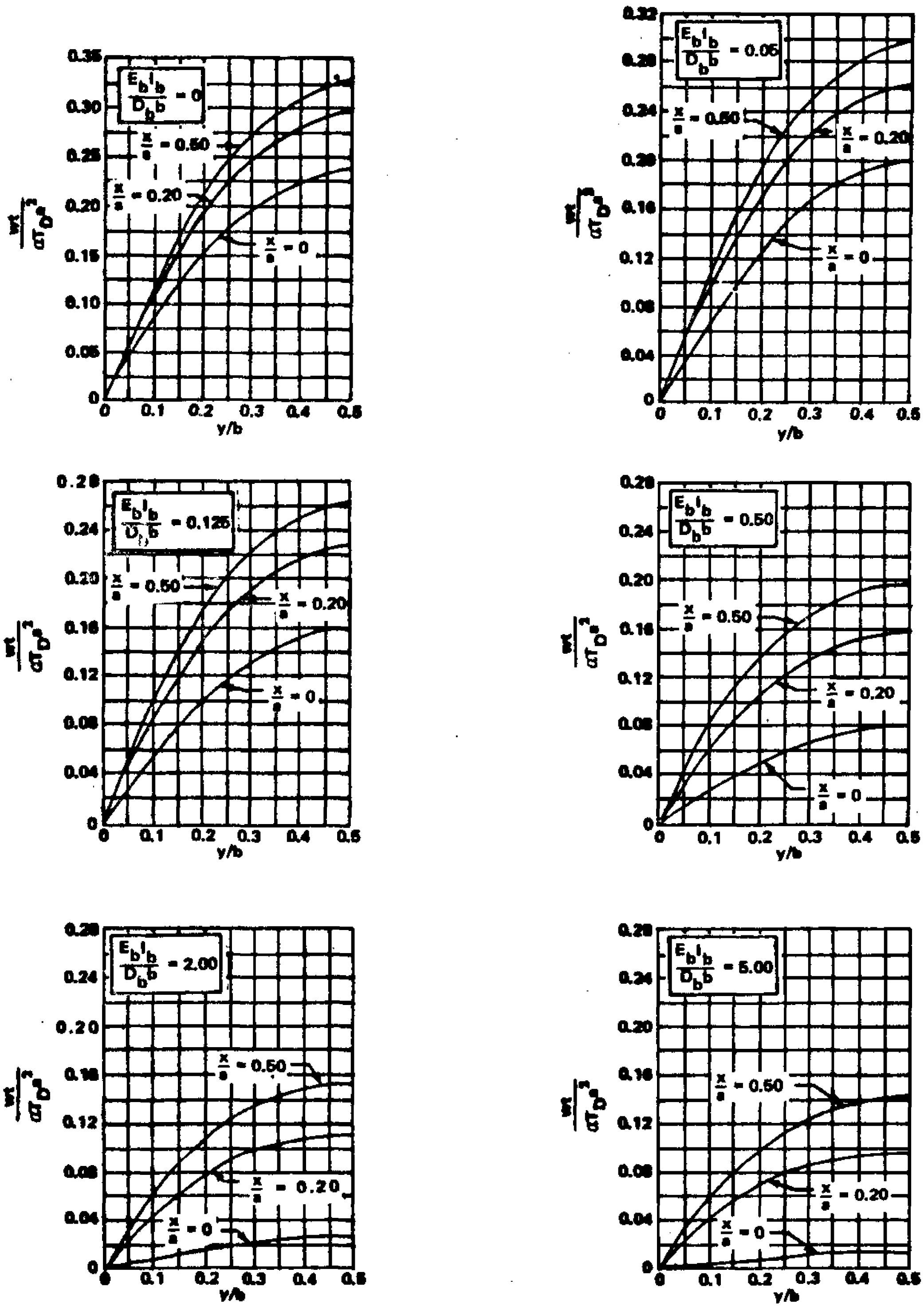

Figure 3.0-30. Nondimensional deflections for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 1.5$, $\nu = 0.30$.

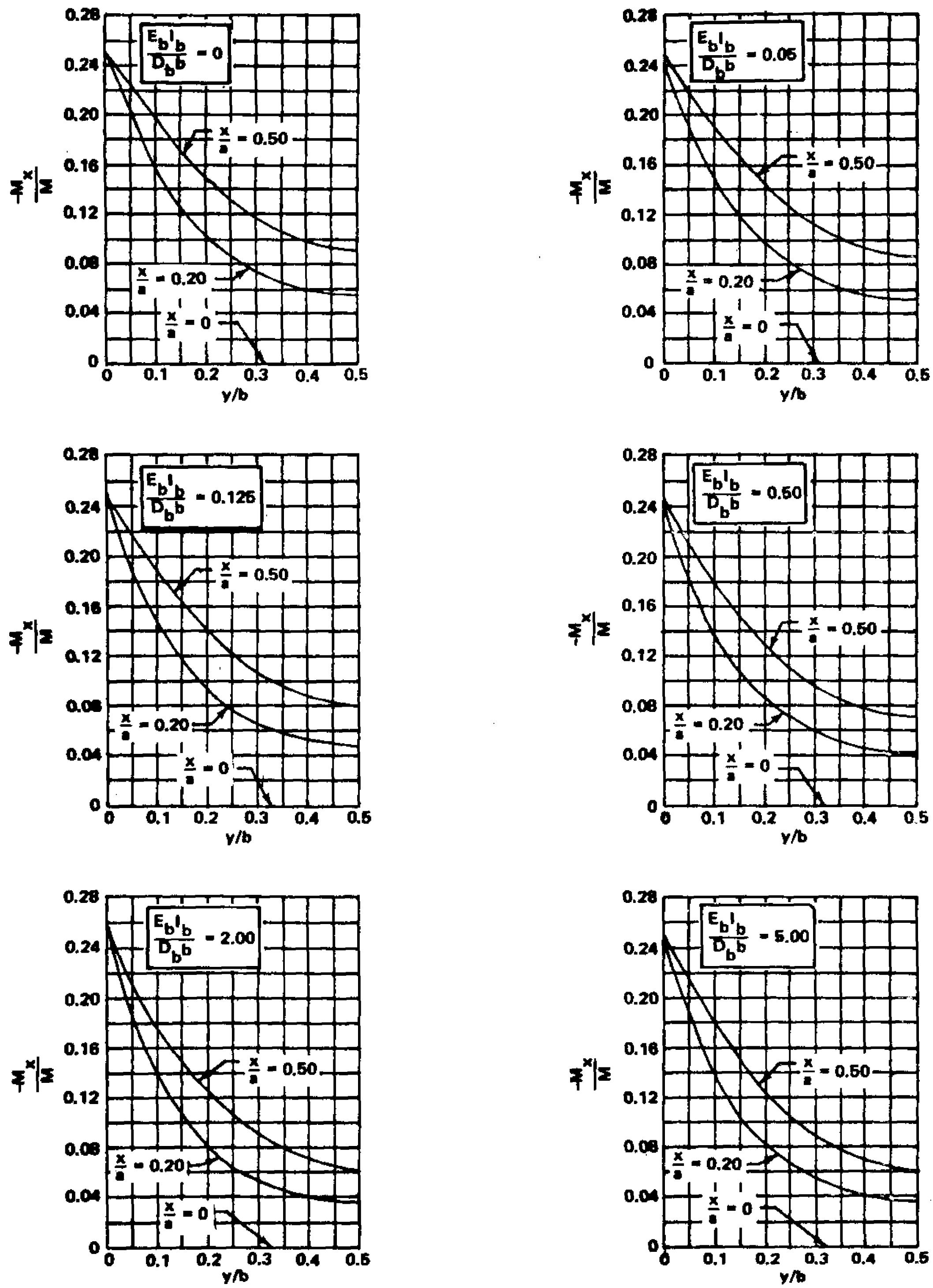

Figure 3.0-31. Nondimensional bending moments M_x/M for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 1.5$, $\nu = 0.30$.

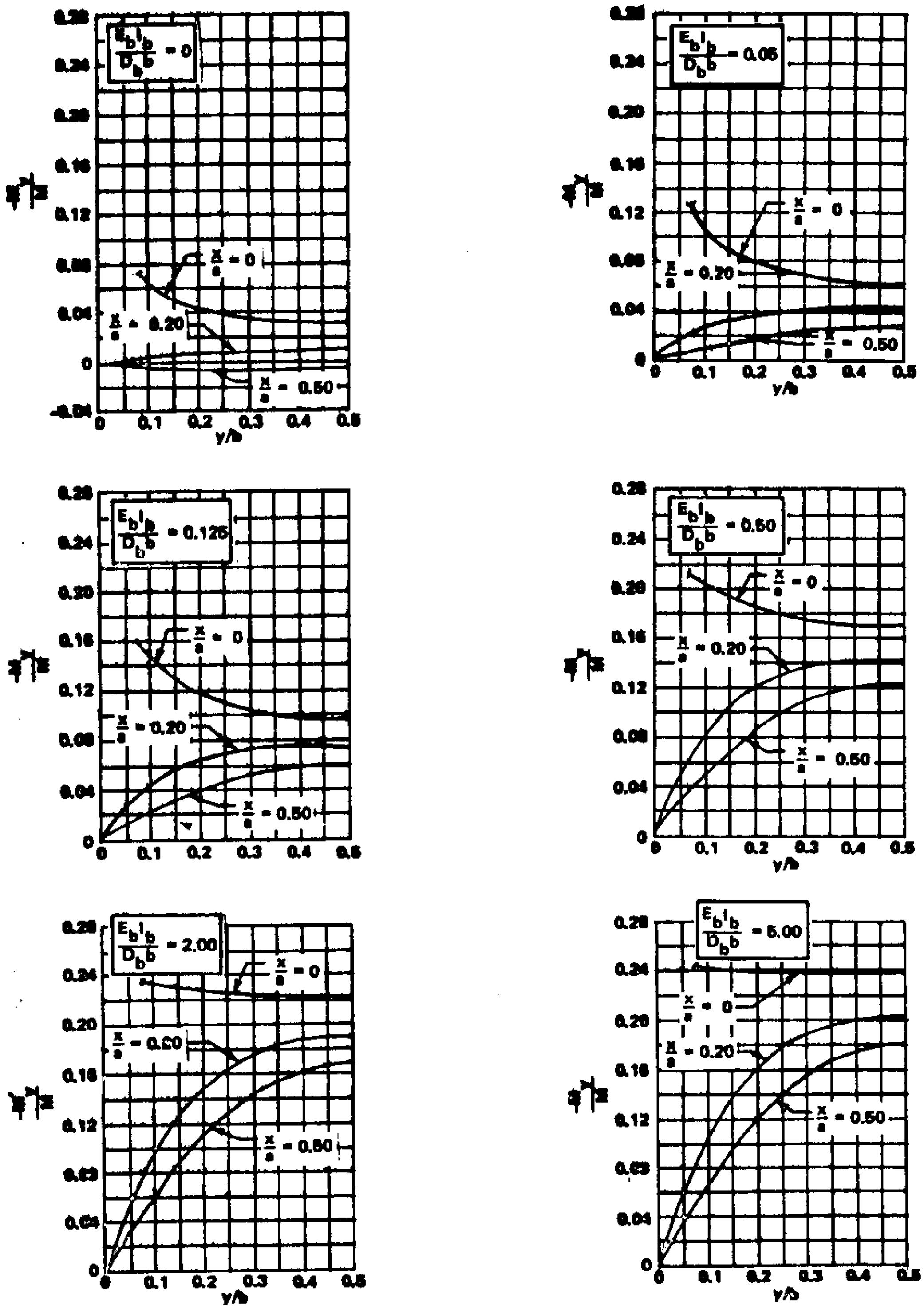

Figure 3.0-32. Nondimensional bending moments M_y/M for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 1.5$, $\nu = 0.30$.

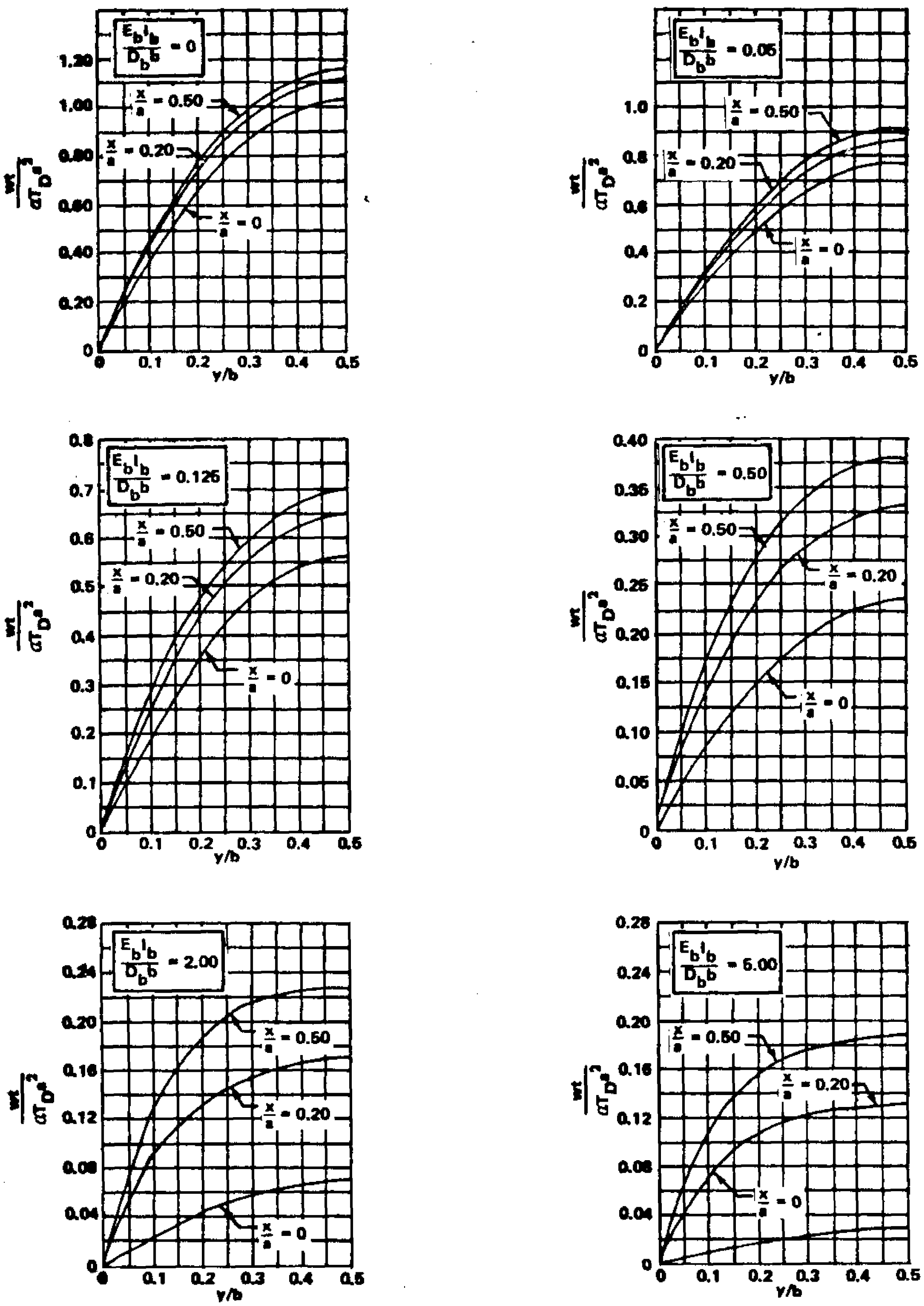

Figure 3.0-33. Nondimensional deflections for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 3.0$, $\nu = 0.30$.

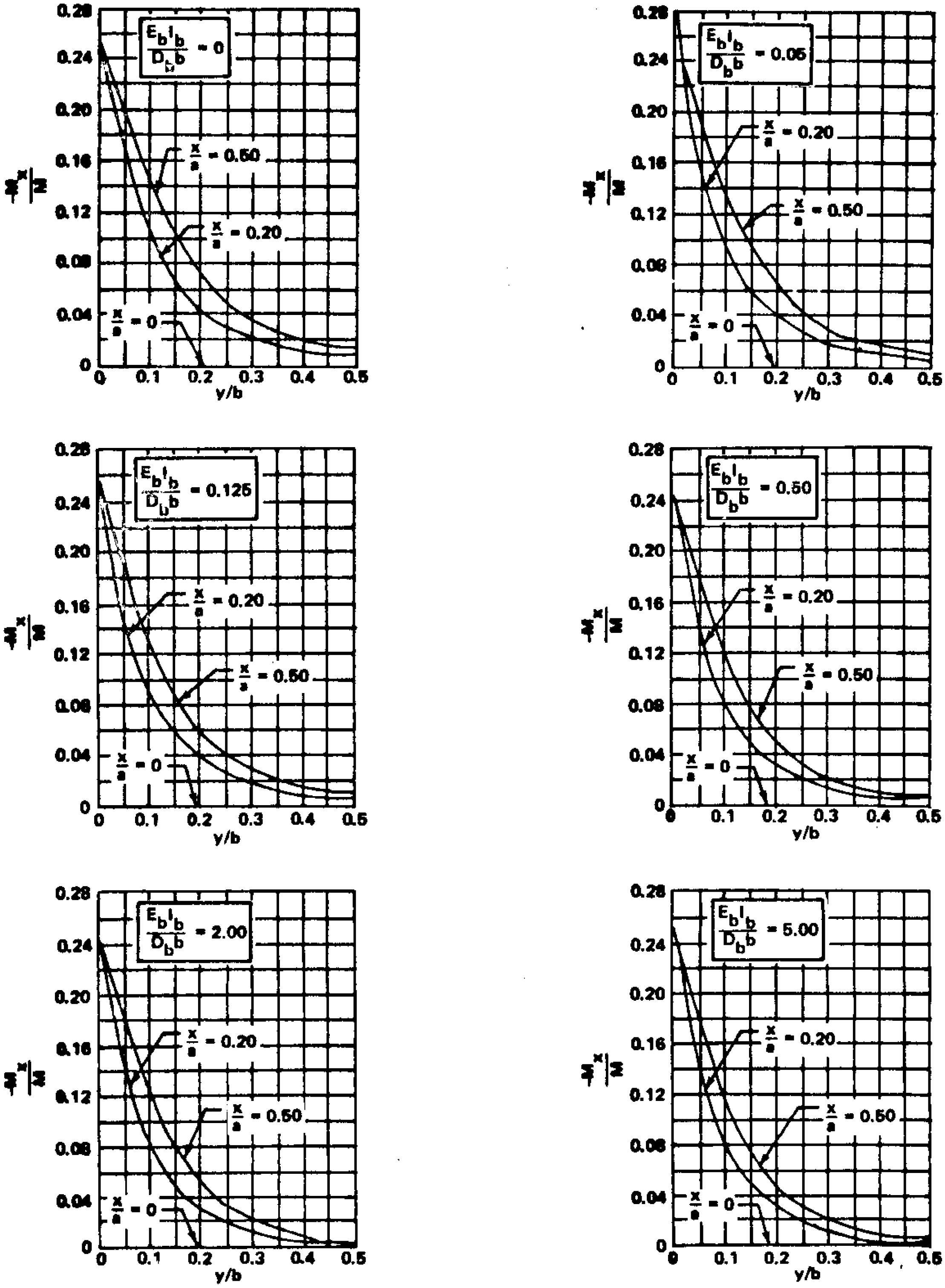

Figure 3.0-34. Nondimensional bending moments M_x/M for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 3.0$, $\nu = 0.30$.

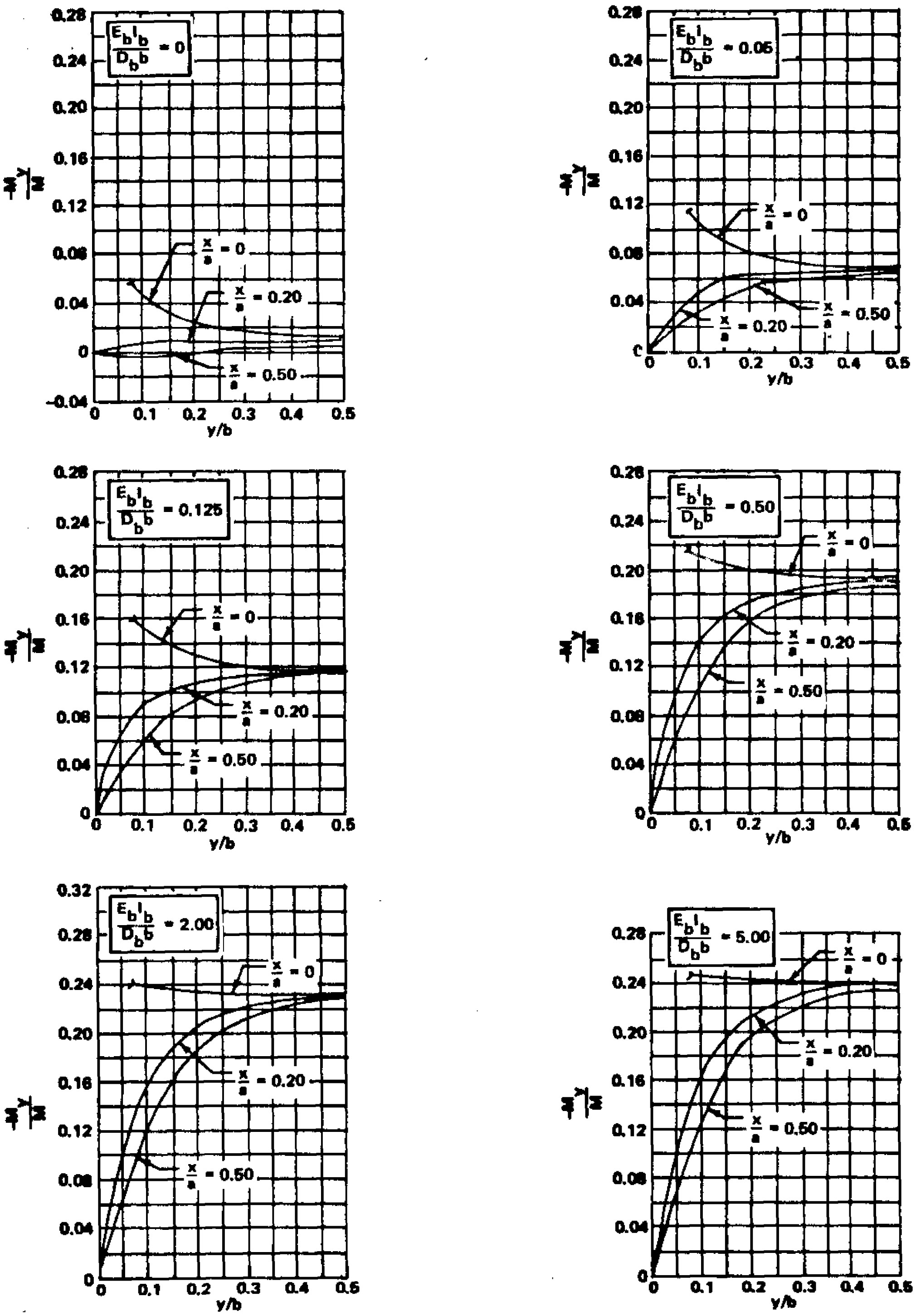

Figure 3.0-35. Nondimensional bending moments M_y/M for a plate with both long edges elastically supported and the short edges simply supported; $b/a = 3.0$, $\nu = 0.30$.

prove profitable to develop a simple digital computer program to perform the summations embodied both in the deflection and moment expressions.

It is important to note that the peak moments always occur at the simply supported boundaries and are oriented so that the corresponding peak values for the stresses σ_x and σ_y act parallel to the edges. The maximum moments can be computed from the following [11]:

$$M_{x(max)} = -\frac{\alpha T_D (1 - \nu^2) D_b}{t} = -E_p t^2 \frac{\alpha T_D}{12} .$$

These moments result from the boundary condition, which demands that $w = 0$ along the simply supported edges. This imposes a straightness constraint that completely suppresses the thermally induced tendency to develop curvatures in vertical planes which pass through these edges.

Arbitrary Temperature Gradient [$T = f(z)$].

The following procedures may be used for the analysis of plates having arbitrary temperature distributions through the thickness $T = f(z)$:

1. The deflections w and bending moments M_x and M_y may be obtained from the equations and figures given for the linear temperature gradient, provided that T_D is replaced by T^*, which may be computed from the following:

$$T^* = \frac{12}{t^2} \int_{-t/2}^{t/2} Tz \, dz .$$

2. The normal stresses x and y may then be established from the relationships

$$\sigma_x = M_x \left(\frac{12z}{t^3}\right) + \frac{E_p \alpha}{(1 - \nu)} \left(\overline{T} + \frac{T^* z}{t} - T\right)$$

and

$$\sigma_y = M_y \left(\frac{12z}{t^3}\right) + \frac{E_p \alpha}{(1 - \nu)} \left(\overline{T} + \frac{T^* z}{t} - T\right)$$

where

$$\overline{T} = \frac{1}{d} \int_{-d/z}^{d/z} T \, dz \quad .$$

II. Temperature Variation Over the Surface.

A. Edges Free or Constrained Against In-Plane Bending.

Configuration.

The design equations provided here apply only to flat, rectangular plates which are of constant thickness and are made of isotropic material. It is assumed that the plate is free of holes and that no stresses exceed the elastic limit. The equations are applicable only for large values of the aspect ratio a/b .

Boundary Conditions.

Consideration is given to each of the following two types of boundary conditions:

1. All edges are free.

2. Plate is fully constrained against in-plane bending but is otherwise completely free.

Temperature Distribution.

The supposition is made that no thermal gradients exist through the plate thickness but a one-dimensional, arbitrary variation occurs over the surface; that is, the temperature is a function only of either x or y.

Design Equations.

It is assumed here that Young's modulus is unaffected by temperature changes. Therefore, in applying the contents of this section, a single effective value must be selected for this property by using some type of averaging technique. On the other hand, the results are presented in a form such that the user may fully account for temperature-dependence of the coefficient of thermal expansion α.

The appropriate stress formulation is developed as follows for the problem which was illustrated in Figure 3.0-26, which shows a rectangular plate with a temperature distribution $T(y)$ and free of any external constraints. The results may be obtained by first imposing a fictitious stress distribution

σ_A on the edges $x = \pm a/2$ such that all thermal deformations are entirely suppressed. It follows that

$$\sigma_A = -\alpha E T(y) \quad .$$

These stresses may be integrated over the width b and the thickness t to arrive at the force

$$P_A = -Et \int_{-b/2}^{b/2} \alpha T(y) \, dy$$

and the moment about the z axis

$$M_A = -Et \int_{-b/2}^{b/2} \alpha T(y) \, y \, dy \quad .$$

Since, at this point in the derivation, it is assumed that no constraints are present, the actual plate must be free of forces and moments on all edges. To restore the plate to such a state, it is necessary to superimpose both a force P_B equal and opposite to P_A and a moment M_B which is equal and opposite to M_A . Hence,

$$P_B = -P_A$$

and

$$M_B = -M_A \quad .$$

The stress corresponding to P_B is easily found to be

$$\sigma_{P_B} = \frac{P_B}{A} = \frac{P_B}{bd} = \frac{E}{b} \int_{-b/2}^{b/2} \alpha T(y)\, dy \quad .$$

The stress corresponding to M_B is

$$\sigma_{M_B} = \frac{M_B y}{I_z} = \frac{12 M_B y}{tb^3} = \frac{12y}{b^3}\, E \int_{-b/2}^{b/2} \alpha T(y)\, y\, dy \quad .$$

It should be noted that the procedure being used constitutes an application of Saint-Venant's principle. Hence, the stresses σ_{P_B} and σ_{M_B} will be accurate representations only at sufficient distances from the edges $x = \pm a/2$. Subject to this limitation, the actual thermal stresses at various points in the plate may be computed from the relationship

$$\sigma = \sigma_A + \sigma_{P_B} + \sigma_{M_B}$$

or

$$\sigma_x = -\alpha E T(y) + \frac{E}{b} \int_{-b/2}^{b/2} \alpha T(y)\, dy + \frac{12y}{b^3}\, E \int_{-b/2}^{b/2} \alpha T(y) y\, dy \quad .$$

The foregoing discussion has been restricted to those cases where the temperature varies only in the y direction. However, the same method can be used to arrive at the following expression when T is a function only of x :

$$\sigma_y = -\alpha ET(x) + \frac{E}{a} \int_{-a/2}^{a/2} \alpha T(x)\, dx + \frac{12x}{a^3} E \int_{-a/2}^{a/2} \alpha T(x) x\, dx \quad .$$

Complex one-dimensional temperature distributions may often be encountered which make it difficult to perform the integrations required by the preceding equations. In such instances, numerical techniques can be used whereby the integral signs are replaced by summation symbols.

The equations were derived for rectangular plates having no edge restraints. However, these relationships can easily be modified to apply when the plate is fully constrained against in-plane bending but is otherwise completely free. This is achieved simply by deleting the final terms from each equation.

Summary of Equations.

1. All Edges Free.

$$T = T(y) \quad ,$$

$$\sigma_x = -\alpha ET(y) + \frac{E}{b} \int_{-b/2}^{b/2} \alpha T(y)\, dy + \frac{12y}{b^3} E \int_{-b/2}^{b/2} \alpha T(y) y\, dy \quad ,$$

$$T = T(x) \quad ,$$

and

$$\sigma_y = -\alpha ET(x) + \frac{E}{a} \int_{-a/2}^{a/2} \alpha T(x)\, dx + \frac{12x}{a^3} E \int_{-a/2}^{a/2} \alpha T(x)\, x\, dx \quad .$$

2. Plate Fully Constrained Against In-Plane Bending But Otherwise Completely Free.

$$T = T(y) \quad ,$$

$$\sigma_x = -\alpha ET(y) + \frac{E}{b} \int_{-b/2}^{b/2} \alpha T(y)\, dy \quad ,$$

$$T = T(x) \quad ,$$

and

$$\sigma_y = -\alpha ET(x) + \frac{E}{a} \int_{-a/2}^{a/2} \alpha T(x)\, dx \quad .$$

B. Edges Fixed.

Configuration.

The equations and sample solution provided here apply only to flat, rectangular plates which are of constant thickness and are made of isotropic material (Fig. 3.0-36). It is assumed that the plate is free of holes and that no stresses exceed the elastic limit. The equations are applicable to any aspect ratio a/b .

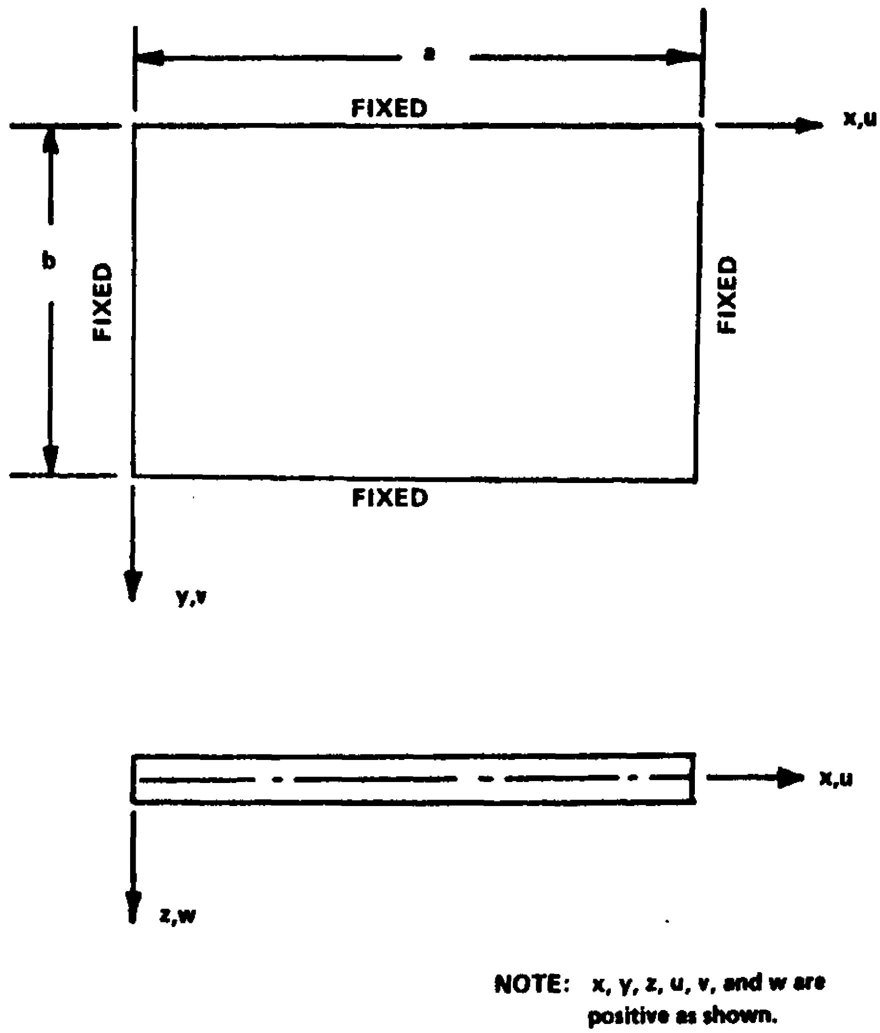

Figure 3.0-36. Rectangular flat plate: all sides fully constrained against in-plane displacements.

Boundary Conditions.

Consideration is given only to plates having all sides completely restrained against in-plane displacements (fixed); that is, both of the following conditions must be satisfied by each edge:

$u = 0$

and

$v = 0$.

Temperature Distribution.

The supposition is made that no thermal gradients exist through the plate thickness. However, temperature variations over the surface may be arbitrary.

Design Equations.

As noted previously, a so-called body-force analogy exists between certain isothermal problems and thermal stress problems for flat plates which experience no transverse displacements w . This method is derived in a number of different references [12, 13, 14] and is frequently referred to as Duhamel's analogy. In Ref. 15, this approach is used to solve the problem being treated here. Assuming that buckling does not occur, solutions in series form were obtained for the in-plane stresses σ_x , σ_y and τ_{xy} . The series coefficients can be obtained by solving the following simultaneous equations:

$$A_{mn} \frac{\pi^2 ab}{4} \left[\left(\frac{m}{a}\right)^2 + \frac{1-\nu}{2} \left(\frac{n}{b}\right)^2 \right] + 2(1+\nu) \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} B_{pq} \frac{mn\, pq}{(p^2 - m^2)(n^2 - q^2)}$$

$$= -\alpha(1+\nu) \int_0^b \int_0^b \frac{\partial T}{\partial x} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

and

$$B_{mn} \frac{\pi^2 ab}{4} \left[\left(\frac{n}{b}\right)^2 + \frac{1-\nu}{2} \left(\frac{m}{a}\right)^2 \right] + 2(1+\nu) \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \frac{mn\,pq}{(p^2-m^2)(n^2-q^2)}$$

$$= -\alpha(1+\nu) \int_0^b \int_0^a \frac{\partial T}{\partial y} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy \quad ,$$

where the indices m, n, p, and q each take on the values 1, 2, 3, . . . subject to the restriction that those values of p and q must be deleted for which $(m \pm p)$ and $(n \pm q)$ are even numbers.

For any given temperature distribution, the right-hand side of the preceding equations must be integrated. In many cases it will be desirable to perform these operations by numerical procedures. The integers m, n, p, and q may be assumed to vary from unity to any value N. This will result in $2N^2$ equations involving N^2 coefficients A_{mn} and N^2 coefficients B_{mn}. This set of equations can be solved simultaneously to determine appropriate values for the coefficients. Once this has been accomplished, the stresses at any point may be computed from

$$\sigma_x = \frac{E}{1-\nu^2} \left(\sum_{m=1}^{N} \sum_{n=1}^{N} \frac{m\pi}{a} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right.$$

$$\left. + \nu \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{n\pi}{b} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \right) - \frac{E\alpha}{1-\nu} T(x,y) \quad ,$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{n\pi}{b}B_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right.$$

$$\left.+\nu\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{m\pi}{a}A_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)-\frac{E\alpha}{1-\nu}T(x,y) \quad ,$$

and

$$\tau_{xy} = G'\sum_{m=1}^{N}\sum_{n=1}^{N}\left(A_{mn}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right.$$

$$\left.+B_{mn}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right) \quad .$$

Then the strains at any point can be determined from

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad ,$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad ,$$

and

$$\gamma_{xy} = \frac{\tau_{xy}}{G'} \quad .$$

Example.

Let a rectangular plate having all four edges fixed (refer to Fig. 3.0-36) be subjected to a temperature gradient

$$T(x, y) = T_0 \left(1 - \frac{x}{a}\right) .$$

Then

$$\frac{\partial T}{\partial x} = -\frac{T_0}{a} \text{ and } \frac{\partial T}{\partial y} = 0 .$$

Substituting these expressions into the right-hand side of the design equation, the following is obtained after integration:

$$A_{mn} \frac{\pi^2 ab}{4} \left[\left(\frac{m}{a}\right)^2 + \frac{1-\nu}{2} \left(\frac{n}{b}\right)^2 \right] + 2(1+\nu) \sum_{p=1}^{N} \sum_{q=1}^{N} B_{pq} \frac{mn\,pq}{(p^2-m^2)(n^2-q^2)}$$

$$= \begin{cases} \dfrac{4(1+\nu)\, b\alpha T_0}{mn\,\pi^2} & \text{if } m \text{ and } n \text{ are odd numbers} \\ 0 & \text{if } m \text{ and } n \text{ are even numbers} \end{cases} ,$$

and

$$B_{mn} \frac{\pi^2 ab}{4} \left[\left(\frac{n}{b}\right)^2 + \frac{1-\nu}{2} \left(\frac{m}{a}\right)^2 \right]$$

$$+ 2(1+\nu) \sum_{p=1}^{N} \sum_{q=1}^{N} A_{pq} \frac{mn\,pq}{(p^2-m^2)(n^2-q^2)} = 0 .$$

Let $N = 2$. Then the preceding equation becomes

$$\frac{\pi^2 ab}{4} A_{11} \left[\left(\frac{1}{a}\right)^2 + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \right] + 2(1+\nu)\, B_{22} \frac{2 \times 2}{(2^2-1)(1-2^2)} = 4 \frac{(1+\nu)\, b\alpha T_0}{\pi^2} ,$$

$$\frac{\pi^2 ab}{4} A_{12} \left[\left(\frac{1}{a}\right)^2 + \frac{1-\nu}{2}\left(\frac{2}{b}\right)^2\right] + 2(1+\nu)\ B_{21}\frac{2\times 2}{(2^2-1)(2^2-1)} = 0 \quad ,$$

$$\frac{\pi^2 ab}{4} A_{21} \left[\left(\frac{2}{a}\right)^2 + \frac{1-\nu}{2}\left(\frac{1}{b}\right)^2\right] + 2(1+\nu)\ B_{12}\frac{2\times 2}{(1-2^2)(1-2^2)} = 0 \quad ,$$

$$\frac{\pi^2 ab}{4} A_{22} \left[\left(\frac{2}{a}\right)^2 + \frac{1-\nu}{2}\left(\frac{2}{b}\right)^2\right] + 2(1+\nu)\ B_{11}\frac{2\times 2}{(1-2^2)(2^2-1)} = 0 \quad ,$$

and

$$\frac{\pi^2 ab}{4} B_{11} \left[\left(\frac{1}{b}\right)^2 + \frac{1-\nu}{2}\left(\frac{1}{a}\right)^2\right] + 2(1+\nu)\ A_{22}\frac{2\times 2}{(2^2-1)(1-2^2)} = 0 \quad ,$$

$$\frac{\pi^2 ab}{4} B_{12} \left[\left(\frac{2}{b}\right)^2 + \frac{1-\nu}{2}\left(\frac{1}{a}\right)^2\right] + 2(1+\nu)\ A_{21}\frac{2\times 2}{(2^2-1)(2^2-1)} = 0 \quad ,$$

$$\frac{\pi^2 ab}{4} B_{21} \left[\left(\frac{1}{b}\right)^2 + \frac{1-\nu}{2}\left(\frac{2}{a}\right)^2\right] + 2(1+\nu)\ A_{12}\frac{2\times 2}{(1-2^2)(1-2^2)} = 0 \quad ,$$

$$\frac{\pi^2 ab}{4} B_{22} \left[\left(\frac{2}{b}\right)^2 + \frac{1-\nu}{2}\left(\frac{2}{a}\right)^2\right] + 2(1+\nu)\ A_{11}\frac{2\times 2}{(1-2^2)(2^2-1)} = 0$$

Solving these equations gives the results,

$$A_{12} = A_{21} = A_{22} = B_{11} = B_{21} = B_{12} = 0 \quad ,$$

while A_{11} and B_{22} can be obtained from the following relationships:

$$A_{11}\frac{\pi^2 ab}{4}\left(\frac{1}{a^2} + \frac{1-\nu}{2}\right) - \frac{8}{9} B_{22} = \frac{4(1+\nu) b\alpha T_0}{\pi^2}$$