C 1.5.0 Torsional Instability of Columns

The critical torsional stress or load for a column is to be determined by use of the following references until this section of the Manual is completed.

1. Argyris, John H., Flexure-Torsion Failure of Panels, Aircraft Engineering, June, 1954.

2. Kappus, Robert, Twisting Failure of Centrally Loaded Open-Section Columns in the Elastic Range, T.M.851, N.A.C.A. 1938.

3. Niles, Alfred S. and J. S. Newell, Airplane Structures, Vol. II Third Edition, John Wiley & Sons, Inc., New York, 1943.

4. Sechler, Ernest E. and L. G. Dunn, Airplane Structural Analysis and Design, John Wiley & Sons, Inc., New York, 1942.

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1. Timoshenko, S. Strength of Materials, Part I and II, Third Edition, D. Van Nostrand Company, Inc., New York, 1957.

2. Seely, Fred B., and J. O. Smith, Advanced Mechanics of Materials, Second Edition, John Wiley & Sons, Inc., New York, 1957.

3. Popov, E. P., Mechanics of Materials, Prentice-Hall, Inc., New York, 1954.

4. Sechler, Ernest E. and L. G. Dunn, Airplane Structural Analysis and Design, John Wiley & Sons, Inc., New York, 1942.

5. Steinbacher, Franz R. and G. Gerard, Aircraft Structural Mechanics, Pitman Publishing Corporation, New York, 1952.

6. Niles, Alfred S. and J. S. Newall, Airplane Structures, Vol. II, Third Edition, John Wiley & Sons, Inc., New York, 1943.

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1. Convair Div. of General Dynamics Corp.

2. Chrysler Corp. Missile Div.

3. North American Aviation

4. Martin

5. Grumman Aircraft

6. McDonnell Aircraft

SECTION C1.5

TORSIONAL INSTABILITY OF COLUMNS

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C1.5.0 TORSIONAL INSTABILITY OF COLUMNS

In the previous sections, it was assumed that the column was torsionally stable; i.e., the column would either fail by bending in a plane of symmetry of the cross section, by crippling, or by a combination of crippling and bending. However, there are cases in which a column will buckle either by twisting or by a combination of bending and twisting. Such torsional buckling failures occur if the torsional rigidity of the section is very low, as for a bar of thin-walled open cross section. Since the difference in behavior of an open cross section is that the torsional rigidity varies roughly as the cube of its wall thickness, thin-walled open sections can buckle by twisting at loads well below the Euler load. Another factor that makes torsional buckling important in thin-walled open sections is the frequent lack of double symmetry. In such sections, centroid and shear center do not coincide and, therefore, torsion and flexure interact.

In this section, it will be assumed that the plane cross sections of the column warp, but their geometric shape does not change during buckling; i.e., the theories consider primary failure of columns as opposed to secondary failure, characterized by distortion of the cross sections.

Separate investigation of primary and local buckling can necessarily give only approximate results because, in general, there will be coupling of primary and secondary buckling. For torsionally stable sections, approximate equations have been developed which include this coupling (Johnson-Euler curves, Section C1.3.2). However, no attempt has been made to formulate a theory which would include coupling of torsion and flexure and local buckling, therefore, an analysis would be extremely complicated.

C1.5.1 CENTRALLY LOADED COLUMNS

Centrally loaded columns can buckle in one of three possible modes: (1) They can bend in the plane of one of the principal axes; (2) they can twist about the shear center axis; or (3) they can bend and twist simultaneously. For any given member, depending on its length and the geometry of its cross section, one of these three modes will be critical. Mode (1) has been discussed in the previous sections. Modes (2) and (3) will be discussed below.

I Two Axes of Symmetry

When the cross section has two axes of symmetry or is point symmetric, the shear center and centroid will coincide. In this case, the purely torsional buckling load about the shear center axis is given by Reference 8.

$$P_\phi = \frac{1}{r_o^2}\left[GJ + \frac{E\Gamma\pi^2}{\ell^2}\right]$$

where:

r_o = polar radius of gyration of the section about its shear center

G = shear modulus of elasticity

J = torsion constant (See Section B8.4.1-IV A)

E = Young's modulus of elasticity

Γ = warping constant of the section (See Section B8.4.1-IV E)

ℓ = effective length of member

Thus, for a cross section with two axes of symmetry there are three critical values of the axial load. They are the flexural buckling loads about the principal axes, P_x and P_y, and the purely torsional buckling load, P_ϕ. Depending on the shape of cross section and length of member, one of these loads will have the lowest value and will determine the mode of buckling. In this case there is no interaction, and the column fails either in pure bending or in pure twisting. Shapes in this category include I-sections, Z-sections, and cruciform sections.

II General Cross Section

In the general case of a column of thin-walled open cross section, buckling occurs by a combination of torsion and bending. Purely flexural or purely torsional buckling cannot occur. To investigate this type of buckling, consider the unsymmetrical cross section shown in Figure C1.5-1. The x and y axes are the principal centroidal axes of the cross section and x_o and y_o are the coordinates of the shear center. During buckling, the cross section will undergo translation and rotation. The translation is defined by the deflections u and v in the x and y directions, respectively, of the shear center o. Thus, during translation of the cross section, point o moves to o' and point c to c'. The rotation of the cross section about the shear center is denoted by the angle ϕ, and the final position of the centroid is c''. Equilibrium of a longitudinal element of a column deformed in this manner leads to three simultaneous differential equations (Reference 8). The solution of these equations yields the following cubic equation for calculating the critical value of buckling load:

$$r_o^2\left(P_{cr}-P_y\right)\left(P_{cr}-P_x\right)\left(P_{cr}-P_\phi\right) - P_{cr}^2 y_o^2\left(P_{cr}-P_x\right) - P_{cr}^2 x_o^2 (P_{cr}-P_y) = 0$$

where

$$P_x = \frac{\pi^2 E I_x}{\ell^2} \;,\quad P_y = \frac{\pi^2 E I_y}{\ell^2} \;,$$

and

$$P_\phi = \frac{1}{r_o^2}\left(GJ + \frac{E\Gamma\pi^2}{\ell^2}\right).$$

Solution of the cubic equation then gives three values of the critical load, P_{cr}, of which the smallest will be used in practical applications. The lowest value of P_{cr} can always be shown to be less than the lowest of the three parameters, P_x, P_y, and P_ϕ. This is to be expected, noting that it represents an interaction of the three individual modes. By use of the effective length, ℓ, various end conditions can be incorporated in the solution above.

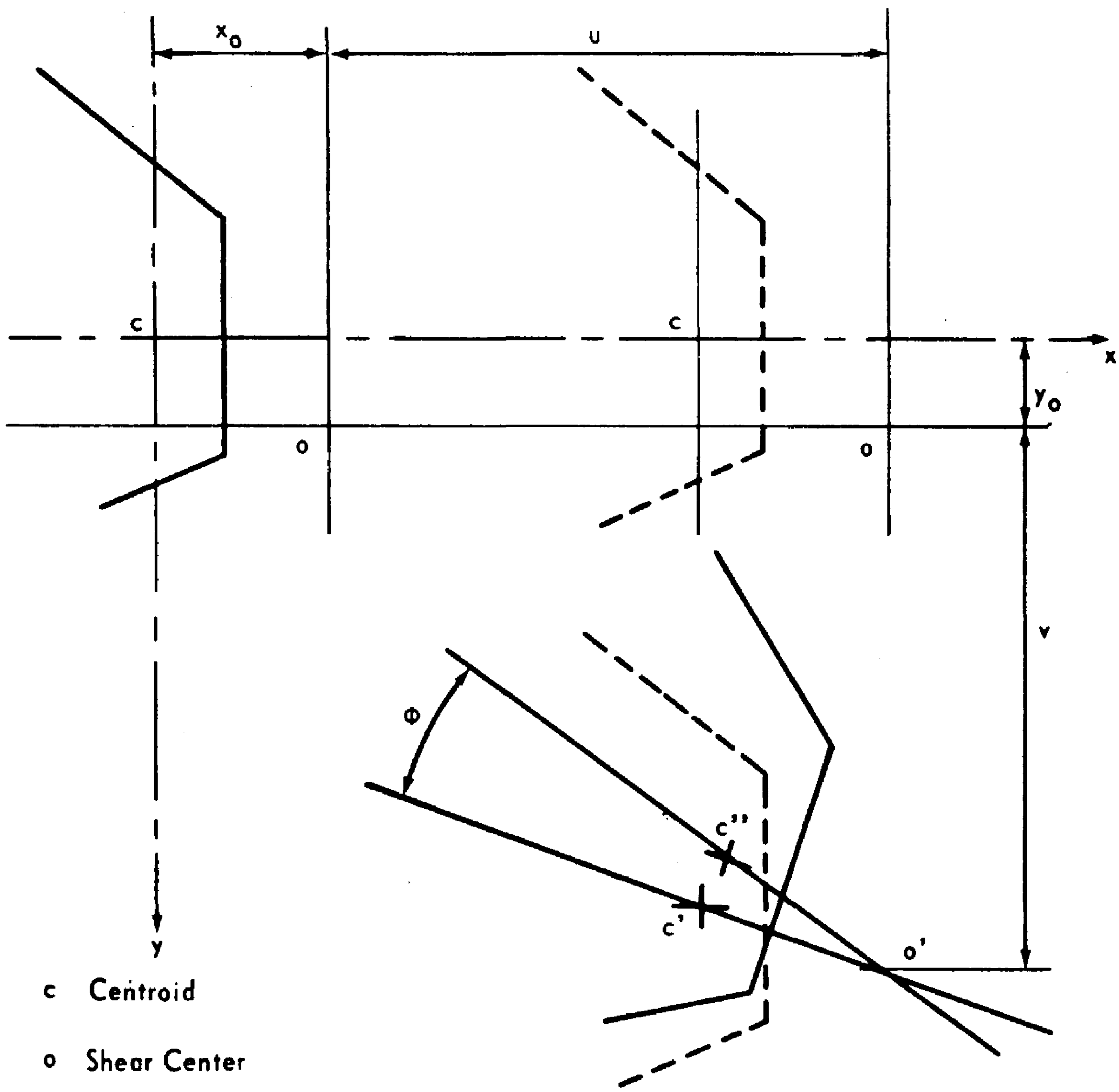


FIGURE C1. 5-1. DISPLACEMENT OF SECTION DURING TORSIONAL — FLEXURAL BUCKLING

III One Axis of Symmetry

If the x-axis is an axis of symmetry, then $y_o = 0$ and the equation for a general section reduces to

$$\left(P_{cr}-P_y\right)\left[r_o^2\left(P_{cr}-P_x\right)\left(P_{cr}-P_\phi\right)-P_{cr}^2 x_o^2\right] = 0 \ . \tag{1}$$

There are again three solutions, one of which is $P_{cr} = P_y$ and represents purely flexural buckling about the y-axis. The other two are the roots of the quadratic term inside the square brackets equated to zero, and give two torsional-flexural buckling loads. The lowest torsional-flexural load will always be below P_x and P_ϕ. It may, however, be above or below P_y. Therefore, a singly symmetrical section (such as an angle, channel, or hat) can buckle in either of two modes, by bending, or in torsional-flexural buckling. Which of these two actually occurs depends on the dimensions and shape of the given section.

The evaluation of the buckling load from equation (1) is often lengthy and tedious. Chajes and Winter (Reference 7) have devised a simple and efficient procedure for evaluating the torsional-flexural buckling load from equation (1) for singly symmetrical sections shown in Figure C1.5-2. In their approach, the essential parameters and their effect on the critical load are clearly evident. Since most shapes used for compression members are singly symmetric, their method is quite useful as described below.

A. Critical Mode of Failure

Failure of singly symmetrical sections can occur either in pure bending or in simultaneous bending and twisting. Because the evaluation of the torsional-flexural buckling load, regardless of the method used, can never be made as simple as the determination of the Euler load, it would be convenient to know if there are certain combinations of dimensions for which torsional-flexural buckling need not be considered at all. To obtain this information, a method of delineating the regions governed by each of the two possible modes of failure has been developed. The method is applicable to any set of boundary conditions. For the purpose of this investigation, however, it will be limited to members with compatible end conditions; i.e., supports that offer equal restraint to bending about the principal axes and to warping.

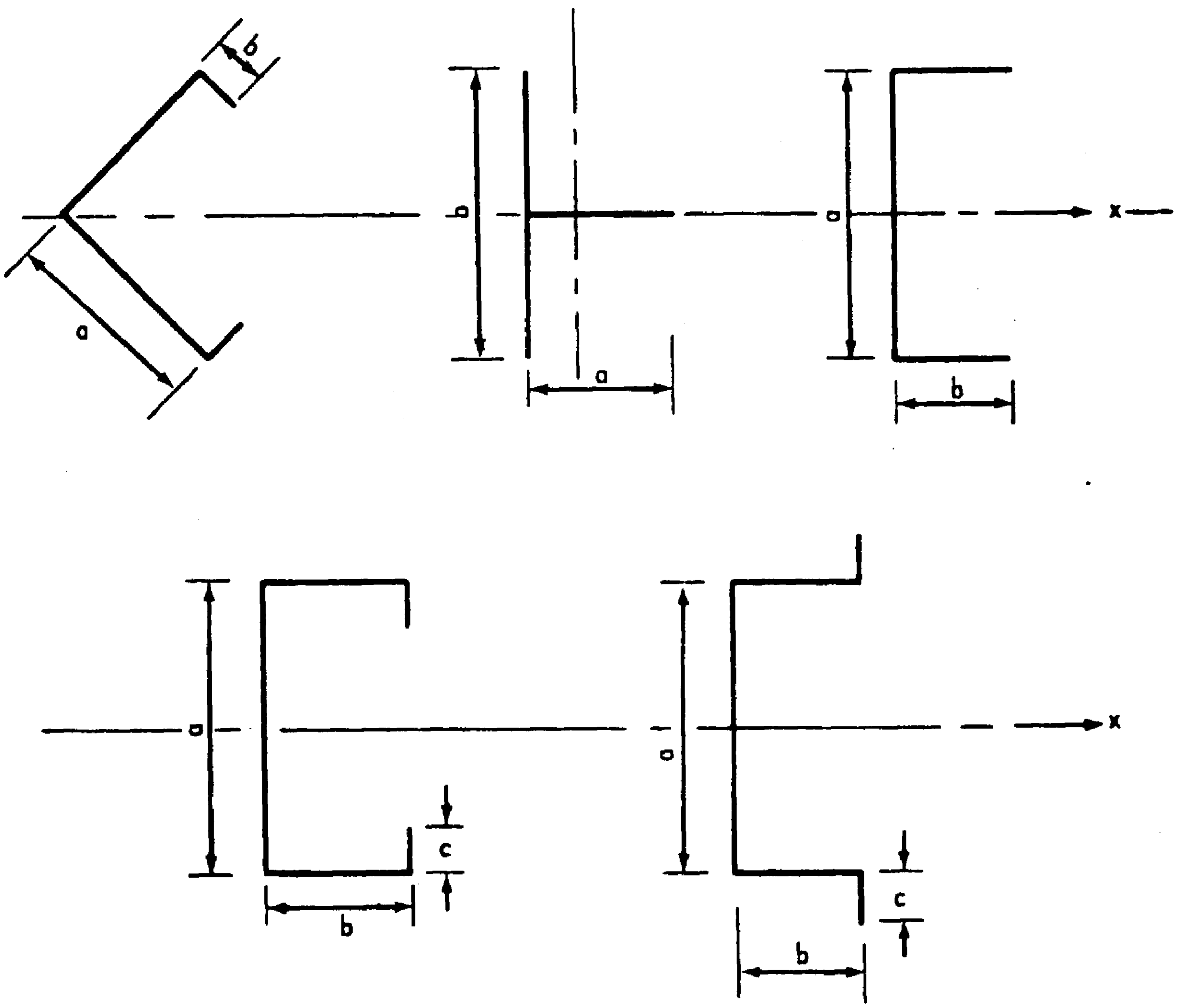


FIGURE C1.5-2 SINGLY SYMMETRICAL SECTIONS

For sections symmetrical about the x axis, the critical buckling load is given by equation (1). According to this equation, the load at which the member actually buckles is either P_y or the smaller root of the quadratic equation, whichever is smaller.

The buckling domain can be visualized as being composed of three regions. These are shown schematically in Figure C1.5-3 for a section whose shape is defined by the width ratio, b/a. Region 1 contains all sections for which $I_y > I_x$.

In this region, only torsional-flexural buckling can occur. Sections for which $I_x > I_y$ fall into Regions 2 or 3. In Region 2, the mode of buckling depends on the parameter $t\ell/a^2$. The $(t\ell/a^2)_{min}$ curve represents the boundary between the two possible modes of failure. It is a plot of the value of $t\ell/a^2$ at which the buckling mode changes from purely flexural to torsional-flexural. The boundary between Regions 2 and 3 is located at the intersection of the $(t\ell/a^2)_{min}$ curve with the b/a axis. Sections in Region 3 will always fail in the flexural mode regardless of the value of $t\ell/a^2$.

Figure C1.5-4 defines these curves for angles, channels, and hat sections. In this figure, members that plot below and to the right of the curve fail in the torsional-flexural mode, whereas those to the left and above fail in the pure bending mode. The curves in Figure C1.5-4 also give the location of the boundaries between the various buckling domains. Each of the curves approaches a vertical asymptote, indicated as a dashed line in the figure. The asymptote, which is the boundary between Regions 1 and 2, is located at b/a corresponding to sections for which $I_x = I_y$. Sections with b/a larger than the transition value at the asymptote will always fail in torsional-flexural buckling, regardless of their other dimensions. If b/a is smaller than the value for the asymptote, then the section falls in Region 2 and failure can be either by pure flexural buckling or in the torsional-flexural mode. In this region, the parameter, $t\ell/a^2$, will determine which of the two possible modes of failure is critical. In the case of the plain and lipped channel section, there is a lower boundary Region 2. This transition occurs where the $(t\ell/a^2)_{lim}$ curve intersects the b/a axis. Sections for which b/a is less than the value at this intersection are located in Region 3. These sections will always fail in the flexural mode, regardless of the value of $t\ell/a^2$. For the lipped angle and hat sections the $(t\ell/a^2)_{lim}$ curve does not intersect the b/a axis. Region 3, where only flexural buckling occurs, does not exist for these sections.

B. Interaction Equation

The critical buckling load for singly symmetrical sections (x-axis is the axis of symmetry) that buckle in the torsional-flexural mode is given by the lowest root of

$$r_o^2\left(P_{cr}-P_x\right)\left(P_{cr}-P_\phi\right) - P_{cr}^2 x_o^2 = 0 \; . \qquad (2)$$

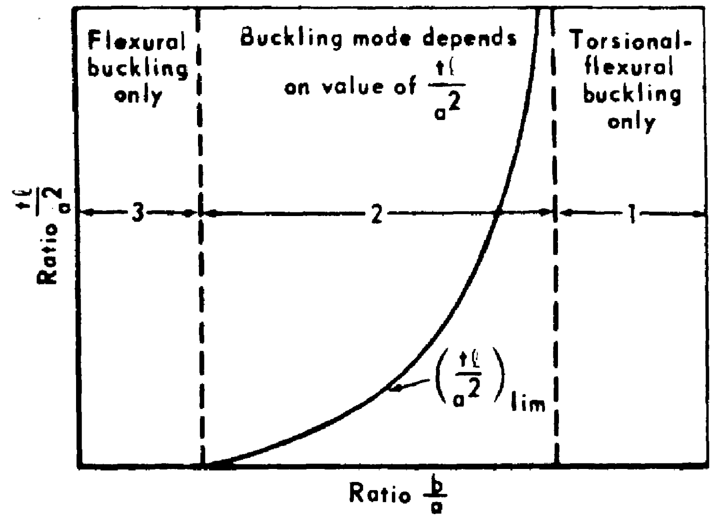


FIGURE C1.5-3 BUCKLING REGIONS

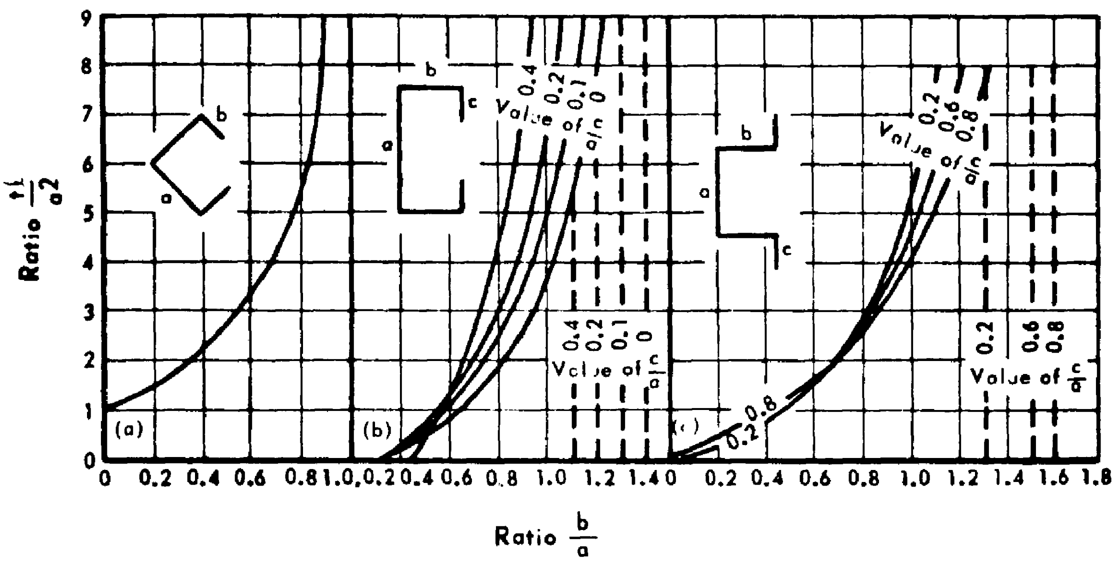


FIGURE C1.5-4 BUCKLING MODE OF SINGLY SYMMETRICAL SECTIONS

Dividing this equation by $P_x P_\phi r_o^2$, and rearranging results in the following interaction equation:

$$\frac{P}{P_\phi} + \frac{P}{P_x} - K\left(\frac{P^2}{P_\phi P_x}\right) = 1 \qquad (3)$$

in which

$$K = 1 - \left(\frac{x_o}{r_o}\right)^2 \qquad (4)$$

is a shape factor that depends on geometrical properties of the cross section.

Figure C1.5-5 is a plot of equation (3). This plot provides a simple method for checking the safety of a column against failure by torsional-flexural buckling.

To determine if a given member can safely carry a certain load, P, it is only necessary to compute P_x and P_ϕ for the section in question and then, knowing K, use the correct curve to check whether the point determined by the arguments P/P_x and P/P_ϕ falls below (safe) or above (unsafe) the pertinent curve. If it is desired to determine the critical load of a member instead of ascertaining whether it can safely carry a given load, use

$$P_{cr} = \frac{1}{2K}\left[\left(P_\phi + P_x\right) - \sqrt{\left(P_\phi + P_x\right)^2 - 4KP_\phi P_x}\right] \qquad (5)$$

which is another form of equation (3).

The interaction equation (eq. 3) indicates that P_{cr} depends on three factors: the loads, P_x and P_ϕ, and the shape factor, K. P_x and P_ϕ are the two factors which interact, while K determines the extent to which they interact. The reason bending and twisting interact is that the shear center and the centroid do not coincide. A decrease in x_o, the distance between these points, therefore causes a decrease in the interaction.

To evaluate the torsional-flexural buckling load by means of the interaction equation, it is necessary to know P_ϕ and K. A convenient method for determining these two parameters is therefore an essential part of the procedure.

C. Evaluation of K

For any given section, K is a function of certain parameters that define the shape of the section. Starting with equation (4) and substituting for

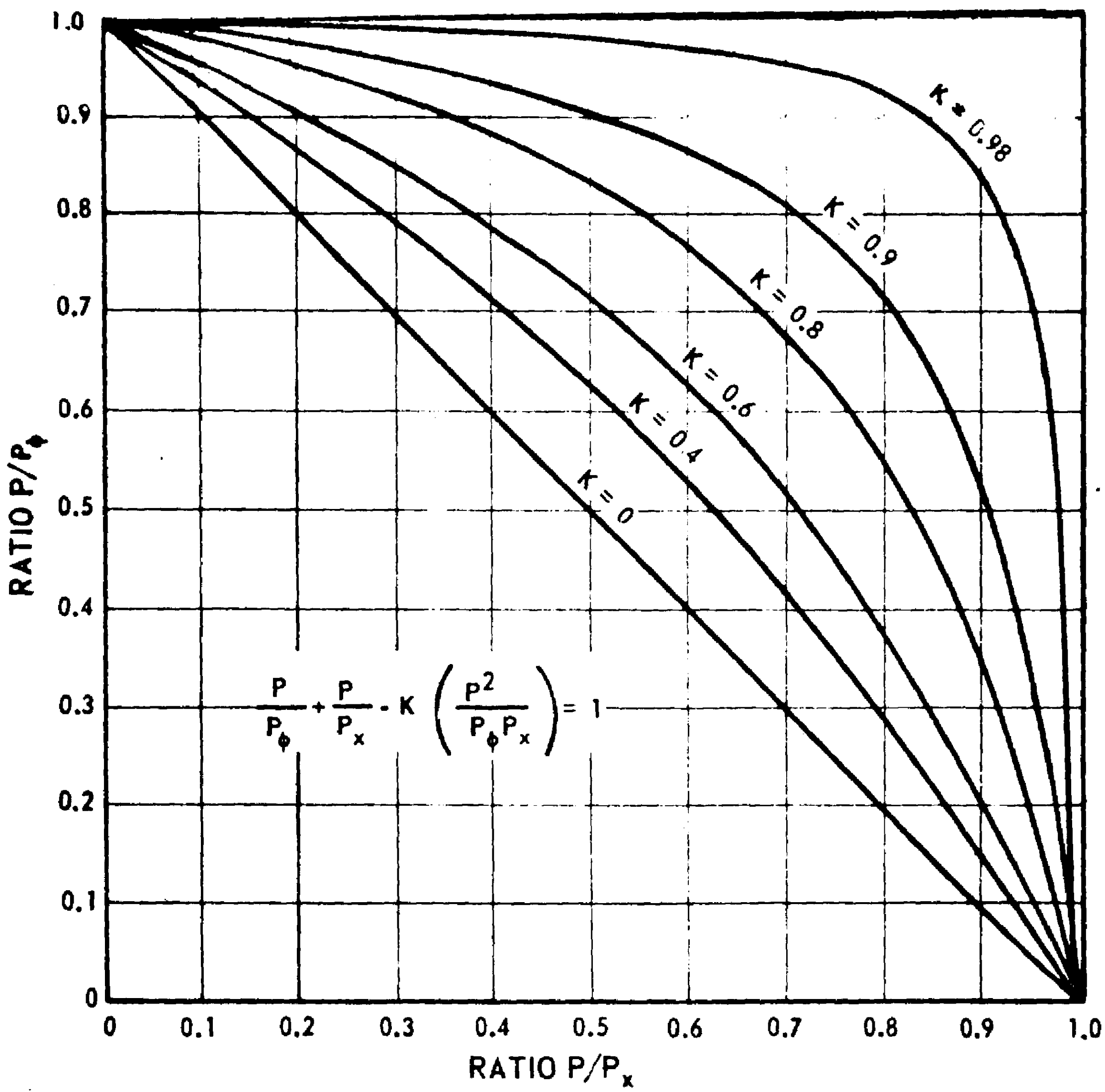


FIGURE C1.5-5 INTERACTION CURVES

x_o and r_o, K can be reduced to an expression in terms of one or more of these parameters. If the thickness of the member is uniform, the parameters will be of the form b/a, in which a and b are the widths of two of the flat components of the section. In the case of a tee section, for example, equation (4) can be reduced to

$$K = 1 - \frac{4}{[1 + b/a]\left[(b/a)^3 + 1\right]} \qquad (6)$$

in which b/a is the ratio of the flange to the leg width (Fig. C1.5-2).

In general, the number of elements of which a section is composed and the number of width ratios required to define its shape will determine the complexity of the relation for K. Because all equal-legged angles without lips have the same shape, K is a constant for this section. For channels and lipped angles, K is a function of a single variable, b/a, while lipped channels and hat sections require two parameters, b/a and c/a, to define K (Fig. C1.5-2).

Curves for determination of K have been obtained for angles, channels, and hat sections. These curves are shown in Figures C1.5-6 and C1.5-7. A single curve covers all equal-legged lipped angle sections. The value of K for all plain equal-legged angles, K = 0.625, is given by the point b/a = 0 on this curve (Fig. C1.5-6). For hats and channels (Figure C1.5-7), a series of curves is given.

D. Evaluation of P_ϕ

The evaluation of P_ϕ follows the same scheme as that used to determine K. Starting with the equation for P_ϕ, given in Paragraph C1.5.1-I, and substituting for r_o, J, and Γ yields

$$P_\phi = EA\left[C_1(t/a)^2 + C_2(a/\ell)^2\right] \tag{7}$$

a general relation for P_ϕ, in which, E = Young's modulus, A = cross-sectional area; t = the thickness of the section; ℓ = effective length of the member; a = the width of one of the elements of the section; and C_1 and C_2 = functions of b/a and c/a, in which b and c are the widths of the remaining elements.

Equation (4) indicates the important parameters in torsional buckling and their effect on the buckling load. Similar to Euler buckling, P_ϕ varies directly with E and A. The term inside the bracket consists of two parts, the St. Venant torsional resistance and the warping resistance. In the first of these, the parameter, t/a, indicates the decrease in torsional resistance with decreasing relative wall thickness; whereas, in the second the parameter a/ℓ shows the decrease in warping resistance with increasing slenderness.

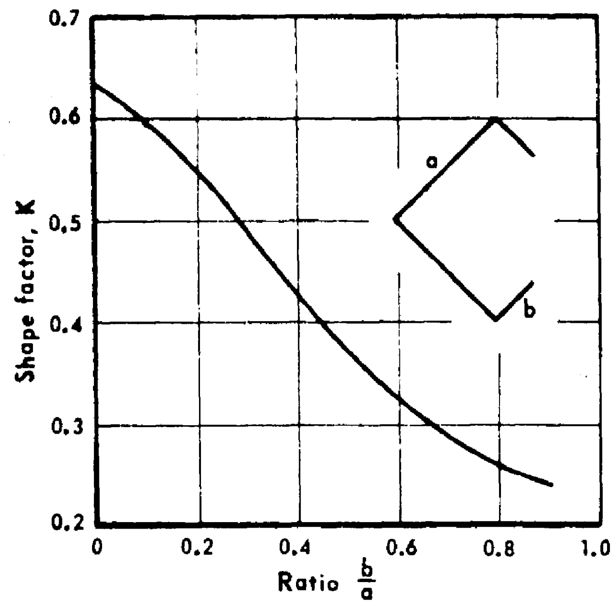


FIGURE C1.5-6 SHAPE FACTOR K FOR EQUAL-LEGGED ANGLES

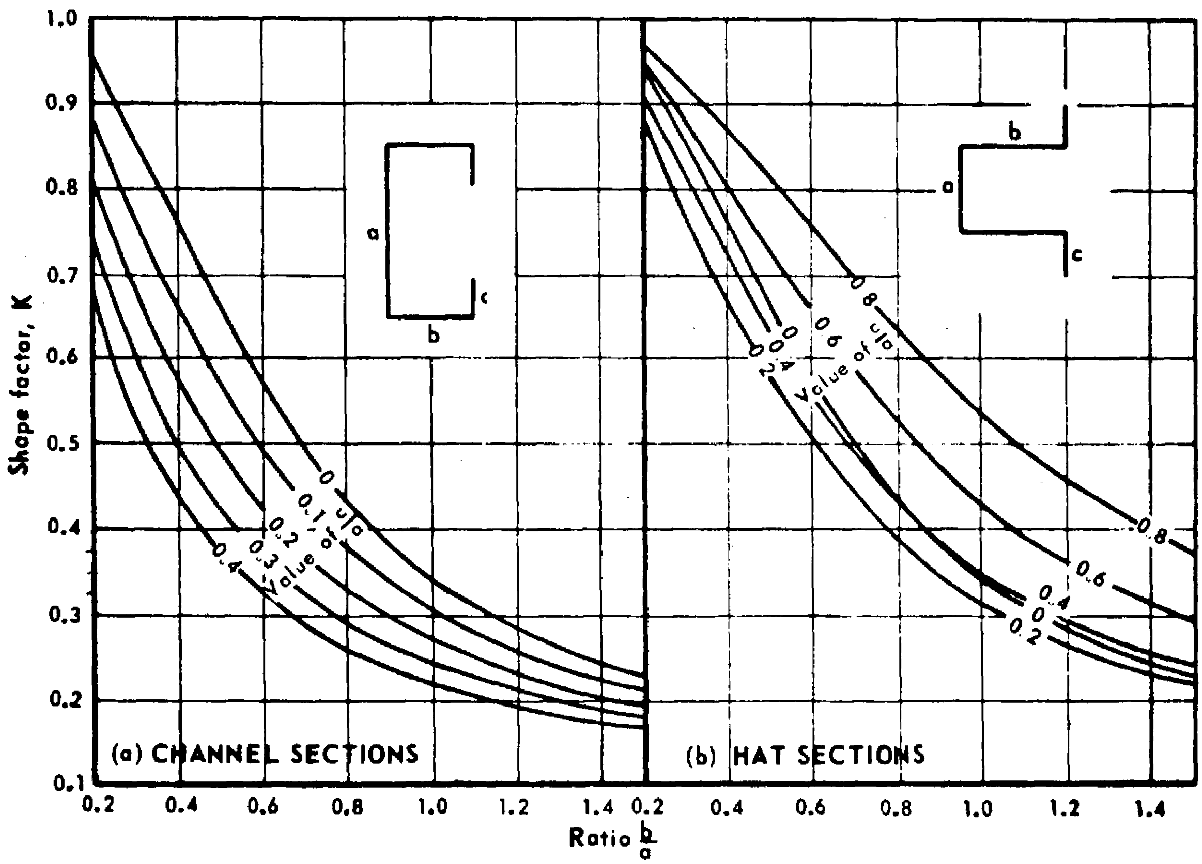


FIGURE C1.5-7 SHAPE FACTOR K

The coefficients, C_1 and C_2, in the St. Venant and warping terms are functions of b/a and c/a, respectively. These terms therefore indicate the effect that the shape of the section has on P_ϕ.

Sections composed of thin rectangluar elements whose middle lines intersect at a common point have negligible warping stiffness; i.e., $\Gamma = 0$. Because C_2 is proportional to Γ, the torsional buckling load of these sections reduces to

$$P_\phi = EAC_1(t/a)^2 \quad . \tag{8}$$

For the plain equal-legged angle, which falls into this category, P_ϕ can be further reduced to

$$P_\phi = AG(t/a)^2 \tag{9}$$

in which G is the shear modulus of elasticity, and a is the length of one of the legs.

In general, however, C_1 and C_2 must be evaluated. Curves for these values are given in Figures C1.5-8, C1.5-9, and C1.5-10 for angles, hats, and channels.

For other cross sections values of the warping constant, Γ, and location of shear center are given in Table I.

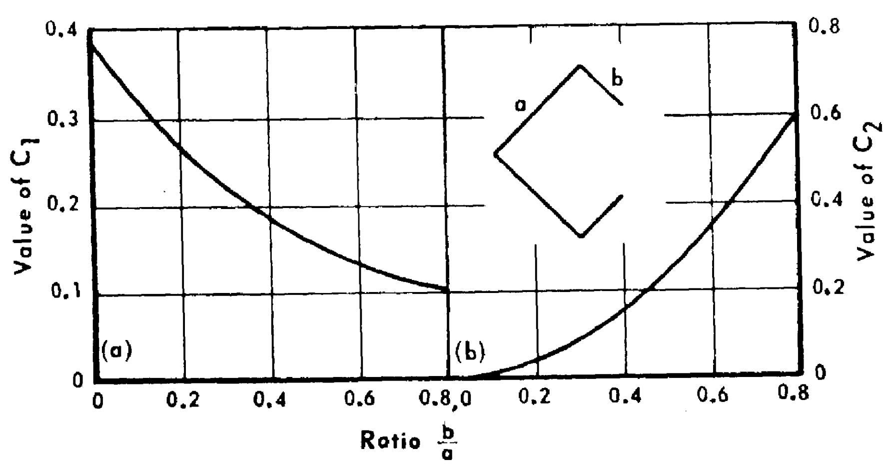


FIGURE C1.5-8 TORSIONAL BUCKLING LOAD COEFFICIENTS, C_1 AND C_2, FOR EQUAL-LEGGED ANGLES

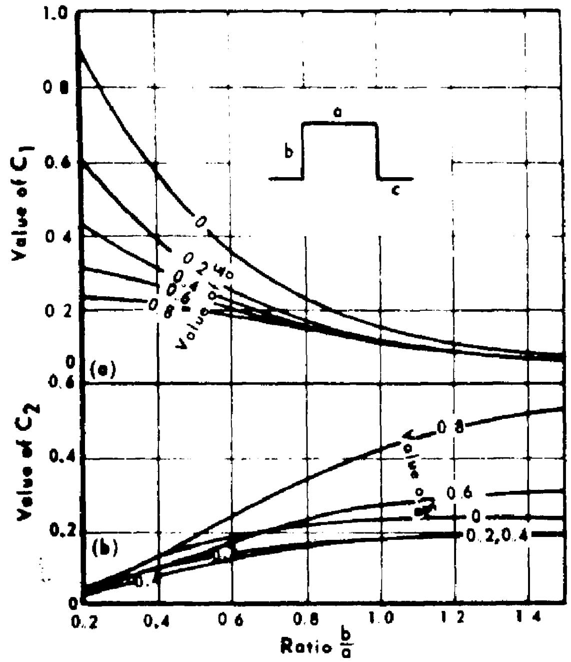


FIGURE C1. 5-9 TORSIONAL BUCKLING LOAD COEFFICIENTS, C_1 AND C_2, FOR HAT SECTIONS

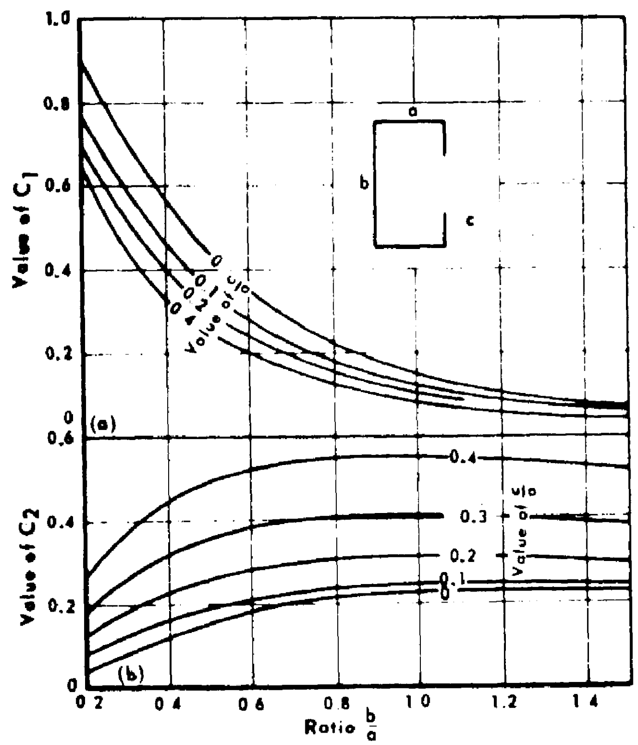
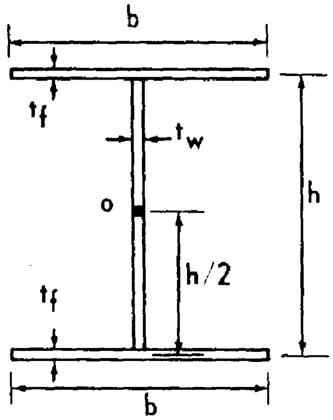


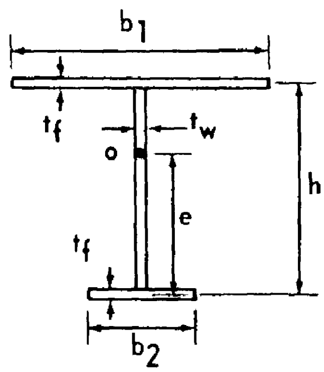
FIGURE C1. 5-10 TORSIONAL BUCKLING LOAD COEFFICIENTS, C_1 AND C_2, FOR CHANNEL SECTIONS

TABLE I. SHEAR CENTER LOCATIONS AND WARPING CONSTANTS FOR VARIOUS CROSS SECTIONS

s = shear center
c = centroid
Γ = warping constant

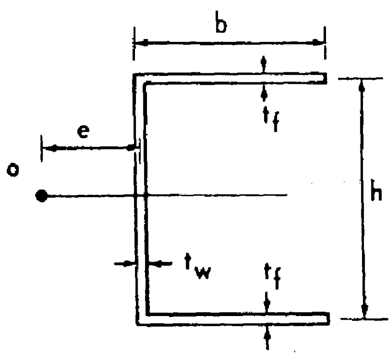


$$\Gamma = \frac{t_f h^2 b^3}{24}$$



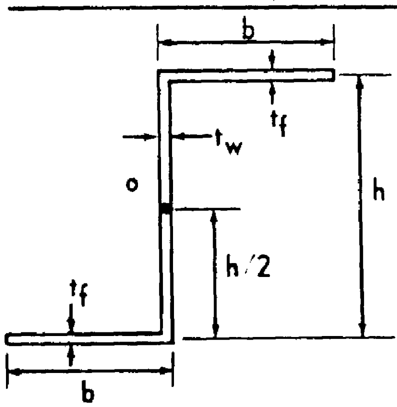
$$e = h\,\frac{b_1^3}{b_1^3 + b_2^3}$$

$$\Gamma = \frac{t_f h^2}{12}\;\frac{b_1^3\, b_2^3}{b_1^3 + b_2^3}$$



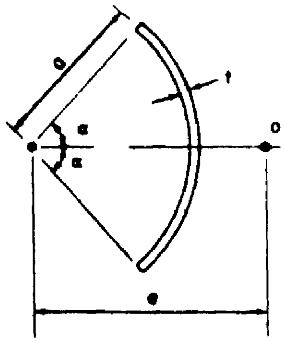
$$e = \frac{3b^2 t_f}{6bt_f + ht_w}$$

$$\Gamma = \frac{t_f b^3 h^2}{12}\;\frac{3bt_f + 2ht_w}{6bt_f + ht_w}$$



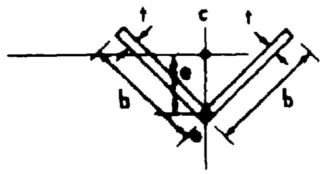
$$\Gamma = \frac{b^3 h^2}{12\,(2b + h)^2}\left[\,2t_f(b^2 + bh + h^2) + 3\,t_w bh\,\right]$$

TABLE I (Continued)



$$e = 2a \frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$$

$$I = \frac{2ta^5}{3}\left[\alpha^3 - \frac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$$



o is located at the intersection of the two legs

$$I = \frac{A^3}{144}$$ A = cross-sectional area of the angle

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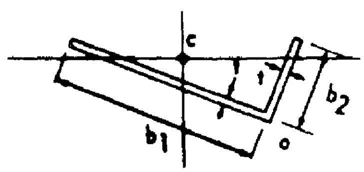
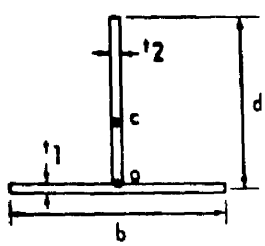


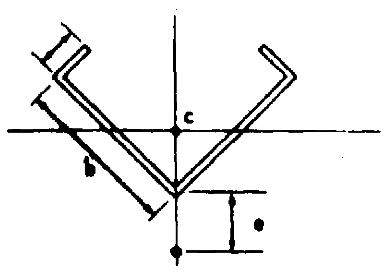
o is located at the intersection of the two legs

$$I = \frac{t^3}{36}(b_1^3 + b_2^3)$$



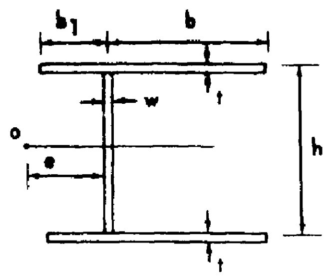
o is located at the intersection of flange and web

$$I = \frac{t_1^3 b}{144} + \frac{t_2^3 d^3}{36}$$

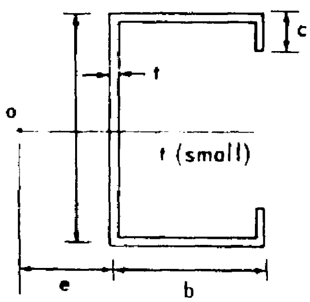


$$e = \frac{b(b_1)^2(3b-2b_1)}{\sqrt{2}\left[2b^3 - (b-b_1)^3\right]}$$

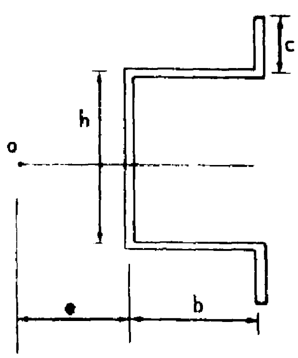
TABLE 1 (Concluded)



$$e = \frac{3(b^2 - b_1^2)}{(w/t)h + 6(b + b_1)} \quad ; \; b_1 < b$$



Values of e/h					
c/h \ b/h	1.0	0.8	0.6	0.4	0.2
0	0.430	0.330	0.236	0.141	0.055
0.1	0.477	0.380	0.280	0.183	0.087
0.2	0.530	0.425	0.325	0.222	0.115
0.3	0.575	0.470	0.365	0.258	0.138
0.4	0.610	0.503	0.394	0.280	0.155
0.5	0.621	0.517	0.405	0.290	0.161



Values of e/h

c/h \ b/h	1.0	0.8	0.6	0.4	0.2
0	0.430	0.330	0.236	0.141	0.055
0.1	0.464	0.367	0.270	0.173	0.080
0.2	0.474	0.377	0.280	0.182	0.090
0.3	0.453	0.358	0.265	0.172	0.085
0.4	0.410	0.320	0.235	0.150	0.072
0.5	0.355	0.275	0.196	0.123	0.056
0.6	0.300	0.225	0.155	0.095	0.040

C1.5.2 SPECIAL CASES

I Continuous Elastic Supports

Consider the stability of a centrally compressed bar which is supported elastically throughout its length and defined by coordinates h_x and h_y (Fig. C1.5-11).

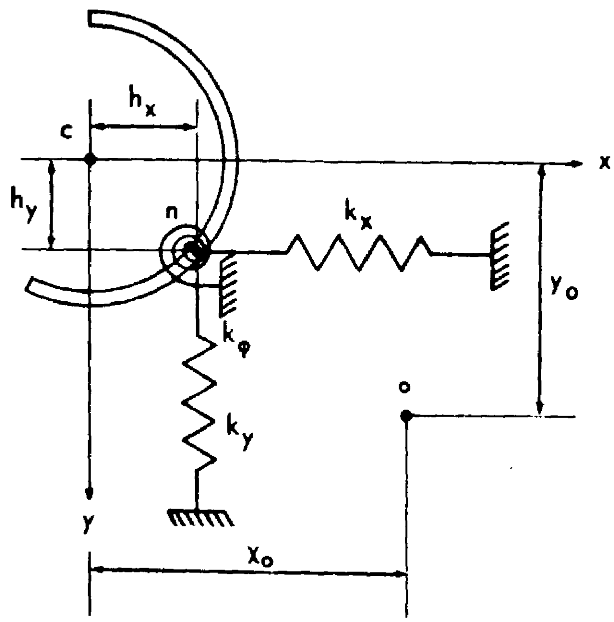


FIGURE C1.5-11 SECTION WITH CONTINUOUS ELASTIC SUPPORTS

For this case, three simultaneous differential equations can be obtained (Reference 8). They are:

$$EI_y \frac{d^4u}{dz^4} + P\left(\frac{d^2u}{dz^2} + y_o \frac{d^2\phi}{dz^2}\right) + k_x\left[u + \left(y_o - h_y\right)\phi\right] = 0$$

$$EI_x \frac{d^4v}{dz^4} + P\left(\frac{d^2v}{dz^2} - x_o \frac{d^2\phi}{dz^2}\right) + k_y\left[v - \left(x_o - h_x\right)\phi\right] = 0$$

$$E\Gamma \frac{d^4\phi}{dz^4} - \left(GJ - \frac{I_o}{A} P\right)\frac{d^2\phi}{dz^2} - P\left(x_o \frac{d^2v}{dz^2} - y_o \frac{d^2u}{dz^2}\right)$$

$$+ k_x\left[u + \left(y_o - h_y\right)\phi\right]\left(y_o - h_y\right) - k_y\left[v - \left(x_o - h_x\right)\phi\right]\left(x_o - h_x\right)$$

$$+ k_\phi \phi = 0 \quad .$$

If the ends of the bar are simply supported, that is, free to warp and to rotate about the x and y axes but with no rotation about the z axis, we can take the solution of the equations above in the form

$$u = A_1 \sin \frac{n\pi z}{\ell} \qquad v = A_2 \sin \frac{n\pi z}{\ell} \qquad \phi = A_3 \sin \frac{n\pi z}{\ell}$$

Substitution of these expressions into the differential equations leads to the evaluation of a determinant and hence to a cubic equation for the critical loads, in the same manner as described in Paragraph C1.5.3-I. The cubic equation is

$$A_3 P^3 + A_2 P^2 + A_1 P + A_0 = 0$$

where

$$A_3 = -\left(\frac{n\pi}{\ell}\right)^6 \frac{\left(I_x + I_y\right)}{A}$$

$$A_2 = E\left[\frac{\left(I_x + I_y\right)^2}{A} + I_y y_o^2 + I_x x_o^2 + \Gamma\right]\left(\frac{n\pi}{\ell}\right)^8 + GJ\left(\frac{n\pi}{\ell}\right)^6 + \left[k_x h_y^2 + k_y h_x^2 + k_\phi + \left(k_x + k_y\right)\frac{\left(I_x + I_y\right)}{A}\right]\left(\frac{n\pi}{\ell}\right)^4$$

$$A_1 = -E^2\left[\frac{I_x I_y I_o}{A} + \left(I_x + I_y\right)\Gamma\right]\left(\frac{n\pi}{\ell}\right)^{10} - EGJ\left(I_x + I_y\right)\left(\frac{n\pi}{\ell}\right)^8$$
$$- E\left[I_y k_x\left(y_o - h_y\right)^2 + I_y k_y\left(x_o - h_x\right)^2 + I_x k_x h_y^2 + I_x k_y h_x^2 + \frac{I_y k_y I_o}{A} + I_x k_x \frac{\left(I_x + I_y\right)}{A} + \left(I_x + I_y\right)k_\phi + \left(I_x + I_y\right)\Gamma\right]\left(\frac{n\pi}{\ell}\right)^6$$
$$- GJ\left(k_x + k_y\right)\left(\frac{n\pi}{\ell}\right)^4 - \left[k_x k_y\left(\frac{I_x + I_y}{A} + h_x^2 + h_y^2\right) + k_\phi\left(k_x + k_y\right)\right]\left(\frac{n\pi}{\ell}\right)^2$$

$$A_0 = E^3 I_x I_y \Gamma\left(\frac{n\pi}{\ell}\right)^{12} + E^2 I_x I_y GJ\left(\frac{n\pi}{\ell}\right)^{10} - E^2\Big[I_x I_y k_x\left(y_o - h_y\right)^2$$

$$+ I_x I_y k_y\left(x_o - h_x\right)^2 + I_x I_y k_\phi + \left(I_y k_y + I_x k_x\right)\Gamma\Big]\left(\frac{n\pi}{\ell}\right)^8$$

$$+ EI_x GJ\left(k_x + k_y\right)\left(\frac{n\pi}{\ell}\right)^6 + E\Big[I_y k_x k_y\left(y_o - h_y\right)^2$$

$$+ I_x k_x k_y\left(x_o - h_x\right)^2 + \left(I_y k_y + I_x k_x\right)k_\phi + k_x k_y \Gamma\Big]\left(\frac{n\pi}{\ell}\right)^4$$

$$+ k_x k_y GJ\left(\frac{n\pi}{\ell}\right)^2 + k_x k_y k_\phi \quad .$$

It can be seen that the values of the coefficients to the cubic equation depend on n. The value of n which minimizes the lowest positive root of the cubic equation must be found. The complexity of this solution may necessitate the use of a computer.

II Prescribed Axis of Rotation

Using the same differential equations given in the previous paragraph, we can investigate buckling of a bar for which the axis is prescribed about which the cross sections rotate during buckling. To obtain a rigid axis of rotation, we have only to assume that $k_x = k_y = \infty$. Then the n axis (Fig. C1.5-11) will remain straight during buckling and the cross sections will rotate with respect to this axis. The resulting differential equation is:

$$\left[E\Gamma + EI_y\left(y_o - h_y\right)^2 + EI_x\left(x_o - h_x\right)^2\right]\frac{d^4\phi}{dz^4}$$

$$- \left[GJ - \frac{I_o P}{A} + P\left(x_o^2 + y_o^2\right) - P\left(h_x^2 + h_y^2\right)\right]\frac{d^2\phi}{dz^2} + k_\phi \phi = 0 \quad .$$

Taking the solution of this equation in the form $\phi = A_3 \sin\frac{n\pi x}{\ell}$

$$P_{cr} = \frac{\left[E\Gamma + EI_y\left(y_o - h_y\right)^2 + EI_x\left(x_o - h_x\right)^2\right]\left(\frac{n\pi}{\ell}\right)^2 + GJ + k_\phi\frac{\ell^2}{n^2\pi^2}}{\frac{I_o}{A} - \left(x_o^2 + y_o^2\right) + \left(h_x^2 + h_y^2\right)} \quad ,$$

we can calculate the critical buckling load in each particular case.

If the bar has two planes of symmetry, the solution is:

$$P_{cr} = \frac{\left(E\Gamma + EI_y h_y^2 + EI_x h_x^2\right)\left(\frac{n^2\pi^2}{\ell^2}\right) + GJ + k_\phi\left(\frac{\ell^2}{n^2\pi^2}\right)}{h_x^2 + h_y^2 + \left(\frac{I_o}{A}\right)} .$$

In each particular case, the value of n which makes P_{cr} a minimum must be found.

If the fixed axis of rotation is the shear-center axis, the solution becomes

$$P_{cr} = \frac{E\Gamma\left(\frac{n^2\pi^2}{\ell^2}\right) + GJ + k_\phi\left(\frac{\ell^2}{n^2\pi^2}\right)}{\frac{I_o}{A}} .$$

This expression is valid for all cross-sectional shapes.

III Prescribed Plane of Deflection

In practical design of columns, the situation arises in which certain fibers of the bar deflect in a known direction during buckling. For example, if a bar is welded to a thin sheet, as in Figure C1.5-12, the fibers of the bar in contact with the sheet cannot deflect in the plane of the sheet. Instead, the fibers along the contact plane nn must deflect only in the direction perpendicular to the sheet. In problems of this type, it is advantageous to take the centroidal axes, x and y, parallel and perpendicular to the sheet. Usually this means that the axes are no longer principal axes of the cross section.

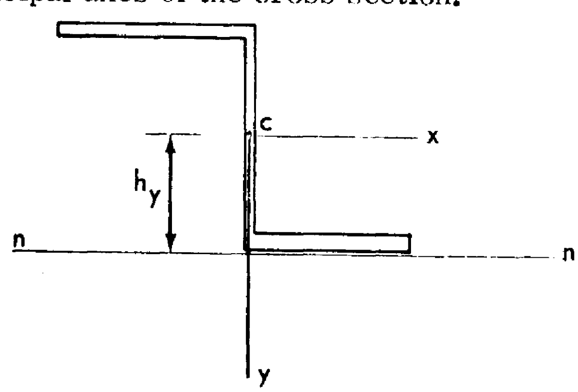


FIGURE C1.5-12 SECTION WITH PRESCRIBED PLANE OF DEFLECTION

For this case, two simultaneous differential equations can be obtained (Reference 8).

They are

$$EI_x \frac{d^4v}{dz^4} + P\frac{d^2v}{dz^2} - EI_{xy}\left(y_o - h_y\right)\frac{d^4\phi}{dz^4} - Px_o \frac{d^2\phi}{dz^2} = 0$$

$$\left[E\Gamma + EI_y\left(y_o - h_y\right)^2\right]\frac{d^4\phi}{dz^4} - \left(GJ - \frac{I_o}{A} P + Py_o{}^2 - Ph_y{}^2\right)\frac{d^2\phi}{dz^2}$$

$$- EI_{xy}\left(y_o - h_y\right)\frac{d^4v}{dz^4} - Px_o \frac{d^2v}{dz^2} = 0$$

These equations can be used to find the critical buckling loads for a given case. As before, taking simple supports and a solution in the form

$$v = A_2 \sin\frac{\pi z}{\ell} \qquad \phi = A_3 \sin\frac{\pi z}{\ell}$$

The following determinant can be obtained:

$$\begin{vmatrix} \left(EI_x \frac{\pi^2}{\ell^2} - P\right) & -\left[EI_{xy}\left(y_o - h_y\right)\frac{\pi^2}{\ell^2} - Px_o\right] \\ \left[-EI_{xy}\left(y_o - h_y\right)\frac{\pi^2}{\ell^2} + Px_o\right] & \left[E\Gamma\frac{\pi^2}{\ell^2} + EI_y\left(y_o - h_y\right)^2\frac{\pi^2}{\ell^2} + GJ - \frac{I_o}{A}P + Py_o{}^2 - Ph_y{}^2\right] \end{vmatrix} = 0$$

From this determinant a quadratic equation for P is obtained from which the critical load can be calculated in each particular case.

$$A_2P^2 + A_1P + A_0 = 0$$

where

$$A_2 = \frac{I_o}{A} - y_o^2 - x_o^2 + h_y^2$$

$$A_1 = -\frac{I_o}{A} P_x + P_x y_o^2 - P_x h_y^2 - \frac{I_o}{A} P_\phi - P_y \left(y_o - h_y\right)^2$$

$$+ 2x_o P_{xy} \left(y_o - h_y\right)$$

$$A_0 = \frac{I_o}{A} P_\phi P_x + P_x P_y \left(y_o - h_y\right)^2 - P^2_{xy} \left(y_o - h_y\right)^2$$

$$P_x = EI_x \frac{\pi^2}{\ell^2}$$

$$P_y = EI_y \frac{\pi^2}{\ell^2}$$

$$P_{xy} = EI_{xy} \frac{\pi^2}{\ell^2}$$

$$P_\phi = \frac{A}{I_o} \left(GJ + E\Gamma \frac{\pi^2}{\ell^2}\right)$$

$$I_o = I_x + I_y + A\left(x_o^2 + y_o^2\right)$$

If the bar is symmetrical with respect to the y axis, as in the case of a channel, the x axis and the y axis become the principal axes. Then, with the substitution of $I_{xy} = 0$ and $x_o = 0$, the two equations become independent. The first of these equations gives the Euler load for buckling in the plane of symmetry. The second equation gives

$$P_{y\phi} = \frac{E\Gamma\left(\frac{\pi^2}{\ell^2}\right) + EI_y \left(y_o - h_y\right)^2 \left(\frac{\pi^2}{\ell^2}\right) + GJ}{\left(\frac{I_o}{A}\right) - y_o^2 + h_y^2}$$

which represents the torsional buckling load for this case.

C1.5.3 ECCENTRICALLY LOADED COLUMNS

I General Cross Section

In the previous sections we have considered the buckling of columns subjected to centrally applied compressive loads only. The case when the force, P, is applied eccentrically (Fig. C1.5-13) will not be considered.

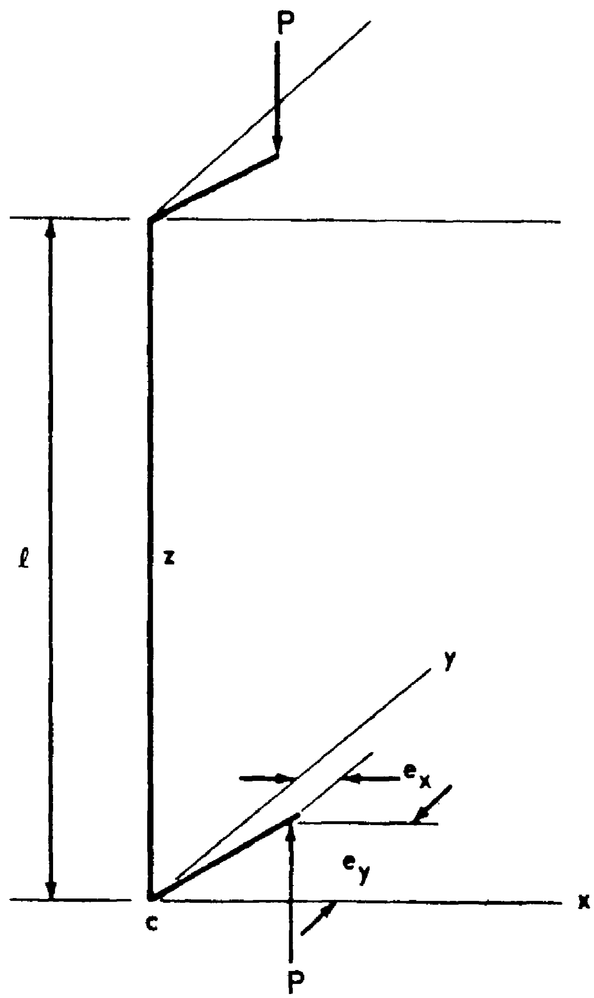


FIGURE C1.5-13 ECCENTRICALLY APPLIED LOAD

In investigating the stability of the deflected form of equilibrium, and considering the case of simply supported ends, the following determinant for calculation of the critical loads is obtained (Reference 8).

$$\begin{vmatrix} \left(P - P_y\right) & 0 & P\left(y_o - e_y\right) \\ 0 & \left(P - P_x\right) & P\left(e_x - x_o\right) \\ P\left(y_o - e_y\right) & P\left(e_x - x_o\right) & Pe_y\beta_1 + Pe_x\beta_2 + P\frac{I_o}{A} - \frac{I_o}{A}P_\phi \end{vmatrix} = 0$$

The solution of this determinant gives the following cubic equation for calculating P_{cr}:

$$A_3P^3 + A_2P^2 + A_1P + A_0 = 0$$

where

$$A_3 = \frac{A}{I_o}\left[e_x\beta_2 + e_y\beta_1 - \left(e_y - y_o\right)^2 - \left(e_x - x_o\right)^2\right] + 1$$

$$A_2 = \frac{A}{I_o}\left[P_x\left(y_o - e_y\right)^2 + P_y\left(x_o - e_x\right)^2 - e_x\beta_2\left(P_x + P_y\right) - e_y\beta_1\left(P_x + P_y\right)\right] - \left(P_x + P_y + P_\phi\right)$$

$$A_1 = \frac{A}{I_o}\left[P_xP_ye_x\beta_2 + P_xP_ye_y\beta_1\right] + \left(P_xP_y + P_yP_\phi + P_xP_\phi\right)$$

$$A_0 = -P_xP_yP_\phi$$

$$P_x = EI_x\frac{\pi^2}{\ell^2}$$

$$P_y = EI_y\frac{\pi^2}{\ell^2}$$

$$P_\phi = \frac{A}{I_o}\left(GJ + E\Gamma\frac{\pi^2}{\ell^2}\right)$$

$$I_o = I_x + I_y + A\left(x_o^{\,2} + y_o^{\,2}\right)$$

$$\beta_1 = \frac{1}{I_x}\left(\int_A y^3 dA + \int_A x^2 y dA\right) - 2y_o$$

$$\beta_2 = \frac{1}{I_y}\left(\int_A x^3 dA + \int_A xy^2 dA\right) - 2x_o \ .$$

In the general case, buckling of the bar occurs by combined bending and torsion. In each particular case, the three roots of the cubic equation can be evaluated numerically for the lowest value of the critical load.

The solution becomes very simple if the thrust, P, acts along the shear-center axis. We then have

$$e_x = x_o \ , \quad e_y = y_o$$

and the buckling loads become independent of each other. In this case, lateral buckling in the two principal planes and torsional buckling may occur independently. Thus, the critical load will be the lowest of the two Euler loads, P_x, P_y, and the load corresponding to purely torsional buckling, which is:

$$P = \frac{\frac{I_o}{A} P_p}{e_y\beta_1 + e_x\beta_2 + \frac{I_o}{A}} \ .$$

II <u>One Axis of Symmetry</u>

Another special case occurs when the bar has one plane of symmetry (which is true for many common sections). Assuming that the yz plane is the plane of symmetry (Fig. C1.5-13), the $x_o = \beta_2 = 0$. The solution for the critical buckling loads is obtained in the same manner as in Paragraph I.

A case of common interest occurs when the load, P, acts in the plane of symmetry; then $e_x = 0$. When this happens, buckling in the plane of symmetry takes place independently and the corresponding critical load is the same as the Euler load. However, lateral buckling in the xz plane and torsional buckling are coupled, and the corresponding critical loads are obtained from the following quadratic equation:

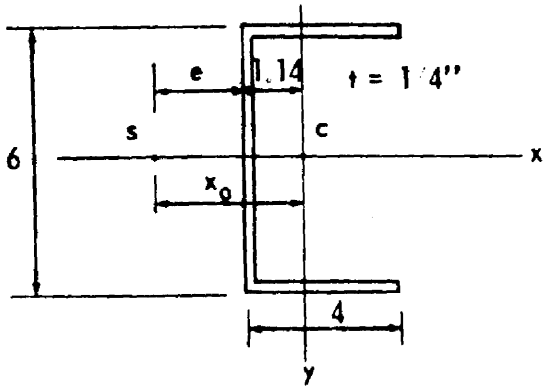
$$\left(P_y - P\right)\left[\frac{I_o}{A} P_\phi - P\left(e_y \beta_1 + \frac{I_o}{A}\right)\right] - P^2\left(y_o - e_y\right)^2 = 0 \quad .$$

III Two Axes of Symmetry

If the cross section of the bar has two axes of symmetry, the shear center and the centroid coincide. Then $y_o = x_o = \beta_1 = \beta_2 = 0$, which simplifies the solutions of Paragraphs I and II somewhat.

C1.5.4 EXAMPLE PROBLEMS FOR TORSIONAL-FLEXURAL INSTABILITY OF COLUMNS

I Example Problem 1



c - centroid

s - shear center

Given:

$L = 60$ in.

$A = 3.5$ in.^{2}

$I_x = 22.5$ in.^{4}

$I_y = 6.05$ in.^{4}

$E = 10.5 \times 10^6$ psi

$G = 4.0 \times 10^6$ psi

$J = .073$ in.^{4}

Find critical load applied at centroid, c, and the mode of buckling. Use general method and also use method of Section C1.5.1-III.

A. Method 1

From Section C1.5.1-III, equation (5),

$$P_{cr} = \frac{1}{2K}\left[\left(P_\phi + P_x\right) - \sqrt{\left(P_\phi + P_x\right)^2 - 4KP_\phi P_x}\right]$$

where $K = 1 - \left(\frac{x_o}{r_o}\right)^2$.

From Table I,

$$e = \frac{3b^2 t_f}{6bt_f + ht_w} = \frac{3(4)^2(\frac{1}{4})}{6(4)(\frac{1}{4}) + 6(\frac{1}{4})} = 1.6 \text{ in.}$$

and

$$\Gamma = \frac{t_f b^3 h^2}{12}\ \frac{3bt_f + 2ht_w}{6bt_f + ht_w}$$

$$\Gamma + \frac{(\frac{1}{4})(4)^3(6)^2}{12}\left[\frac{3(4)(\frac{1}{4}) + 2(6)(\frac{1}{4})}{6(4)(\frac{1}{4}) + 6(\frac{1}{4})}\right]$$

$$\Gamma = 38.4 \text{ in.}^6$$

$$I = I_x + I_y + A X_o^2$$

$$I_o = 22.5 + 6.05 + 3.5(2.74)^2$$

$$I_o = 54.75 \text{ in.}^4$$

$$r_o^2 = \frac{I_o}{A} = \frac{54.75}{3.5} = 15.65 \text{ in.}^2$$

$$P_x = \frac{\pi^2 E I_x}{\ell^2} = \frac{\pi^2 10.5 \times 10^6 \times 22.5}{(60)^2} = 647,691 \text{ lbs}$$

$$P_y = \frac{\pi^2 E J_y}{\ell^2} = \frac{\pi^2 10.5 \times 10^6 \times 6.05}{(60)^2} = 174,000 \text{ lbs}$$

$$P_\phi = \frac{1}{r_o^2}\left[GJ + \frac{E\Gamma\pi^2}{\ell}\right]$$

$$P_\phi = \frac{1}{15.65}\left[4 \times 10^6(.073) + \frac{10.5 \times 10^6(38.4)\pi^2}{(60)^2}\right]$$

$$P_\phi = 89,200 \text{ lbs}$$

$$K = 1 - \left(\frac{x_o}{r_o}\right)^2 = 1 - \frac{(2.74)^2}{15.65} = 0.55$$

$$P_{cr} = \frac{1}{2(0.55)}\Big[(89,200 + 647,691) -$$

$$\sqrt{(89,200 + 647,691)^2 - 4(0.55)(89,200)(647,691)}\Big]$$

$$P_{cr} = 82,727 \text{ lbs} \ ,$$

therefore, critical load is 82,727 pounds and the mode is torsional-flexural buckling.

B. Method 2

Check Figure C1.5-4(b) for critical mode of buckling with

$$b/a = 4/6 = 0.66 \ , \quad c/a = 0 \ , \quad \frac{t\ell}{a^2} = 0.416 \ .$$

Since the point plots are below the curve for $c/a = 0$, the critical mode of buckling will be torsional-flexural buckling.

From Figure C1.5-7(a), $K = 0.53$

From Figure C1.5-10, $C_1 = 0.31$ and $C_2 = 0.20$.

From Section C1.5.1-III, equation (7),

$$P_\phi = EA\left[C_1\,(t/a)^2 + C_2(a/\ell)^2\right]$$

$$P_\phi = 10.5 \times 10^6(3.5)\left[0.31\left(\frac{0.25}{6}\right)^2 + 0.2\left(\frac{6}{60}\right)^2\right]$$

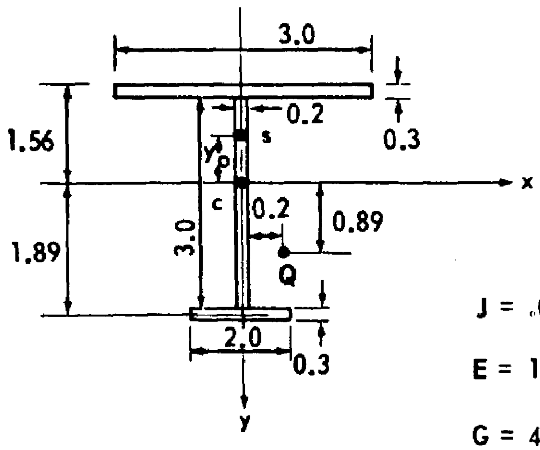
$$P_\phi = 93,500 \text{ lbs}$$

$$P_x = \frac{\pi^2 EI_x}{\ell^2} = 647,691 \text{ lbs}$$

$$P_{cr} = \frac{1}{2(0.53)}\left[(93,500 + 647,691) - \sqrt{(93,500 + 647,691) - 4(0.53)(93,500)(647,691)}\right]$$

$P_{cr} = 86,668$ lbs

II Example Problem 2



c = centroid

s = shear center

Q = point of load application

Given: $A = 2.1$ in.^{2}

$J = .053$ in.^{4} $\quad$ $I_x = 4.43$ in.^{4}

$E = 10.5 \times 10^6$ psi $\quad$ $I_y = .88$ in.^{4}

$G = 4.0 \times 10^6$ psi $\quad$ $L = 50$ in.

Find: Critical load applied at point Q.

To locate shear center and evaluate the warping constant, refer to Table I.

$$e = \frac{hb_1^3}{b_1^3 + b_2^3} = \frac{3.3(3.0)^3}{(3.0)^3 + (2.0)^3} = -2.546$$

$$y_o = -2.546 + 1.89 = -0.656$$

$$\Gamma = \frac{t_f h^2}{12} \frac{b_1^3 b_2^3}{b_1^3 + b_2^3}$$

$$\Gamma = \frac{0.3(3.3)^2}{12} \frac{(3)^3(2)^3}{(3)^3 + (2)^3}$$

$$\Gamma = 1.68 \text{ in.}^6$$

To calculate I_o, β_1, β_2 refer to Paragraph C1.5.3-I:

$$I_o = I_x + I_y \; A(y_o)^2$$

$$I_o = 4.43 + 0.88 + (2.1)(-0.656)^2$$

$$I_o = 6.21 \text{ in.}^4$$

$$\beta_1 = \frac{1}{I_x}\left(\int_A y^3 da + \int x^2 y da\right) - 2y_o$$

$$\beta_1 = \frac{1}{4.43}\left[(3.0)(0.3)(-1.41)^3 + 3.0(0.2)(0.24)^3 + 2.0(0.3)(1.89)^3\right]$$

$$-2(-0.656)$$

$$\beta_1 = 1.66 \text{ in.}$$

$$\beta_2 = 0$$

$$P_x = \frac{\pi^2 E I_x}{\ell^2} = \frac{\pi^2 10.5 \times 10^6 (4.43)}{(50)^2} \quad 183,633 \text{ lbs}$$

$$P_y = \frac{\pi^2 E I_y}{\ell^2} \quad 36,477 \text{ lbs}$$

$$P_\varphi = \frac{A}{I_o}\left[GJ + \frac{E\Gamma\pi^2}{\ell^2}\right] = \frac{2.1}{6.21}\left[1 \times 10^6(0.053) + \frac{10.5 \times 10^6(1.68)\pi^2}{(50)^2}\right]$$

$$P_\varphi \quad 95,239 \text{ lbs} .$$

Now calculate the coefficients to the cubic equation in Paragraph C1.5.3-I:

$$A_3 = \frac{A_o}{I_o}\left[e_x\beta_2 + e_y\beta_1 - \left(e_y - y_o\right)^2 - \left(e_x - y_o\right)^2\right] + 1$$

$$A_3 = \frac{2.1}{6.21}\left[(0.2)(0) + (0.89)(1.66) - (0.89 + 0.655)^2 - (0.2-0)^2\right] + 1$$

$$A3 = 0.6789$$

$$A_2 = \frac{A}{I_o}\left[P_x\left(y_o - e_y\right)^2 + P_y\left(x_o - e_x\right)^2 - e_x\beta_2\left(P_x + P_y\right) - e_y\beta_1\left(P_x + P_y\right)\right]$$

$$-\left(P_x + P_y + P_\varphi\right)$$

$$A_2 = \frac{2.1}{6.21}\left[183,633(-0.655-0.89)^2 + 36,477(0-0.2)^2\right.$$

$$\left.-0.2(0)(183,633 + 36,477) - (0.89)(1.66)(183,633 + 36,477)\right]$$

$$A_2 = -276,596$$

$$A_1 = \frac{A}{I_o}\left[P_x P_y e_x \beta_2 + P_x P_y e_y \beta_1\right] + \left(P_x P_y + P_y P_\phi + P_x P_\varphi\right)$$

$$A_1 = \frac{2.1}{6.21}\left[183,633(36,477)(0.2)(0) + 183,633(36,477)(0.89)(1.66)\right]$$

$$+ \left[183,633(36,477) + 36,477(95,394) + (183,633)(95,394)\right]$$

$$A_1 = 31.042 \times 10^9$$

$$A_0 = - P_x P_y P_\varphi$$

$$A_0 = - (183,633)(36,477)(95,394)$$

$$A_0 = - 638.985 \times 10^{12}$$

$$A_3P^3 + A_2P^2 + A_1P + A_0 = 0 \quad .$$

Dividing by A_3, $P^3 + \frac{A_2}{A_3}P + \frac{A_1}{A_3}P + \frac{A_0}{A_3} = 0$

let $k = \frac{A_2}{A_3}$, $q = \frac{A_1}{A_3}$. $r = \frac{A_0}{A_3}$

$$k = -4.0742 \times 10^5$$

$$q = 4.5725 \times 10^{10}$$

$$r = -9.4123 \times 10^{14} \quad .$$

For solution of cubic, let $P = X - k/3$, then the cubic reduces to

$$X^3 + aX + b = 0$$

where

$$a = 1/3(3q - k^2)$$

$$b = \frac{1}{27}(2k^3 - 9kq + 27r)$$

$$a = -0.9605 \times 10^{10}$$

$$b = 0.259 \times 10^{15}$$

$$Q = \frac{b^2}{4} + \frac{a^3}{27} = -0.01605 \times 10^{30} \quad .$$

Since $Q < 0$, we have three real, unequal roots given by

$$X_k = -2\sqrt{-\frac{a}{3}} \cos\left(\frac{\phi}{3} + 120K\right) \quad , \quad K \quad 0, 1, 2$$

where

$$\phi = \arc\cos\sqrt{\frac{b^2}{4} \div \left(-\frac{a^3}{27}\right)} \qquad \phi = 45°39'$$

$$X_o = -109,200$$

$$X_1 = 80,324$$

$$X_2 = 28,876 \quad .$$

Substitute these roots into $P = x - k/3$ for critical load values.

$$P_1 = X_1 - 1/3$$

$$P_1 = -109,200 - \frac{(-4.0742 \times 10^5)}{3}$$

$$P_1 = 26,606 \text{ lbs}$$

$$P_2 = 80,324 + 135,806$$

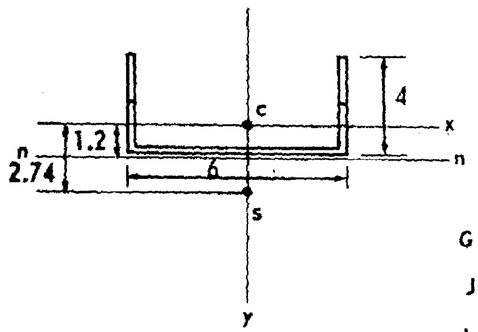
$$P_2 = 216,130 \text{ lbs}$$

$$P_3 = 28,876 + 135,806$$

$$P_3 = 164,682 \text{ lbs} \quad .$$

Therefore, the critical load is $P_1 - 26,606$ pounds.

III Example Problem 3



c - centroid

s - shear center

$\Gamma = 38.4 \text{ in.}^6$

$A = 2.1 \text{ in.}^2$

$G = 4 \times 10^6$ psi

$I_{xx} = 6.05 \text{ in.}^4$

$J = .073$

$I_y = 22.5 \text{ in.}^4$

$L = 60$ in.

$E = 10.5 \times 10^6$ psi

Find the critical load applied at point c (centroid) with prescribed deflection normal to plane n-n (refer to Section C1.5.2-III). For Euler buckling about the x axis the critical load is

$$P_x = \frac{\pi^2 E I_x}{\ell^2} = \frac{\pi^2 10.5 \times 10^6 (6.05)}{(60)^2} = 174,157 \text{ lbs} \quad .$$

The torsional buckling load is

$$P_{y\phi} = \frac{E\Gamma\left(\frac{\pi^2}{\ell^2}\right) E I_y \left(y_o - h_y\right)^2 \left(\frac{\pi^2}{\ell^2}\right) + GJ}{\frac{I_o}{A} - y_o^{\,2} + h_y^{\,2}}$$

$$P_{y\phi} = \frac{10.5 \times 10^6 (38.4)\left(\frac{\pi}{60}\right)^2 + 10.5 \times 10^6 (22.5)(2.74 - 1.2)^2 \left(\frac{\pi}{60}\right)^2 + 4 \times 10^6 (0.073)}{\frac{54.75}{2.1} - (2.74)^2 + (1.2)^2}$$

$$P_{y\phi} = 146,672 \text{ lbs} \quad .$$

Therefore, the critical load is $P_{y\phi}$ (146,672 lbs) and the mode of failure is torsional buckling, assuming no local failures.

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