SECTION B4

BEAMS

TABLE OF CONTENTS

B 4.0.0 BEAMS

B 4.1.0 Simple Beams

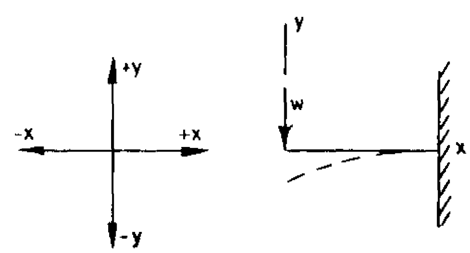
B 4.1.1 Shear, Moment, and Deflection

The general equations relating load, shear, bending moment, and deflection are given in Table B 4.1.1.1. These equations are given in terms of deflection and bending moments.

Title	Y	M
Deflection	$\triangle = y$	$\triangle = \iint \frac{M}{EI} dx\ dx$
Slope	$\theta = dy/dx$	$\theta = \int \frac{M}{EI} dx$
Bending Moment	$M = EI\ d^2y/dx^2$	M
Shear	$V = EI\ d^3y/dx^3$	$V = dM/dx$
Load	$W = EI\ d^4y/dx^4$	$W = dv/dx = d^2M/dx^2$

Table B 4.1.1.1

Sign Convention



a) x is positive to the right.
b) y is positive upward.
c) M is positive when the compressed fibers are at the top.
d) W is positive in the direction of negative y.
e) V is positive when the part of the beam to the left of the section tends to move upward under the action of the resultant of the vertical forces.

The limiting assumptions are:

a) The material follows Hooke's Law.
b) Plane cross sections remain plane.
c) Shear deflections are negligible.
d) The deflections are small.

B 4.1.1 Shear, Moment and Deflection (Cont'd)

The deflection of short, deep beams due to vertical shear may need to be considered. The differential equation of the deflection curve including the effects of shearing deformation is:

$$y = \int\int \frac{M\,dxdx}{EI} + \int \frac{KV}{AG}\,dx$$

(K) is the ratio of the maximum shearing stress on the cross section to the average shearing stress. The value of (K) is given by the equation:

$$K = \frac{A}{I\ b} \int_{o}^{a} b'ydy$$

(I) is the moment of inertia of the cross-section with respect to the centrodial axis and (a), (b), (b'), and (y) are the dimensions shown in Fig. B 4.1.1-1.
(A) is the area of the cross-section

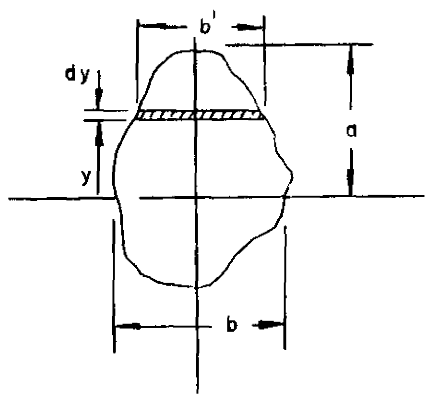


Fig. B 4.1.1-1

Table B 4.1.1-2 Beam Formulas

Notation: W = load (lb); w = unit load (lb. per linear in.); M is positive when clockwise; V is positive when acting upward; y is positive when upward. Constraining moments, applied couples, loads, and reactions are positive when acting as shown. All forces are in pounds, all moments in inch-pounds; all deflections and dimensions in inches. θ is in radians and tan $\theta = \theta$.

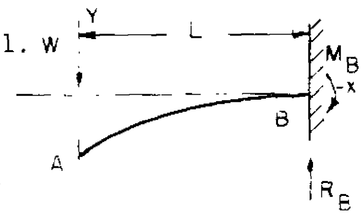
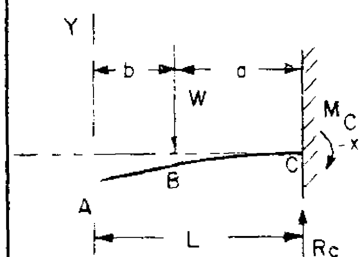
<table>
<tr><th colspan="2">Cantilever Beams</th></tr>
<tr><th>Type of loading and Case number</th><th>Reactions, Vertical Shear, Bending Moments, Deflection y, and Slope</th></tr>
<tr><td>1. W

</td><td>$R_B = +W$; $V = -W$
$M_x = -Wx$; Max $M = -WL$ at B
$y = -\frac{1}{6}\frac{W}{EI}(x^3 - 3L^2x + 2L^3)$; Max $y = -\frac{1}{3}\frac{WL^3}{EI}$ at A; $\theta = \frac{1}{2}\frac{WL^2}{EI}$ at A</td></tr>
<tr><td>2.

</td><td>$R_C = +W$; (A to B) $V = 0$; (B to C) $V = -W$
(A to B) $M = 0$; (B to C) $M = -W(x-b)$; Max $M = -Wa$ at C
(A to B) $y = -\frac{1}{6}\frac{W}{EI}(-a^3 + 3a^2L - 3a^2x)$;
(B to C) $y = -\frac{1}{6}\frac{W}{EI}\left[(x-b)^3 - 3a^2(x-b) + 2a^3\right]$;
Max $y = -\frac{1}{6}\frac{W}{EI}(3a^2L - a^3)$; $\theta = \frac{1}{2}\frac{Wa^2}{EI}$ (A to B)</td></tr>
</table>

Table 4.1.1-2 Beam Formulas (Cont'd)

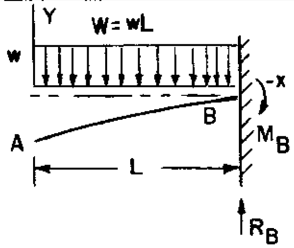
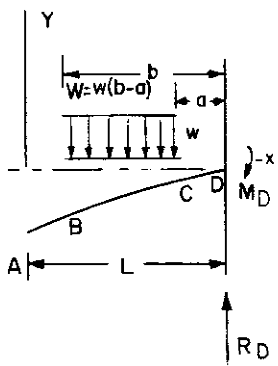
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
3. 	$R_B = +W;\ V = -\frac{W}{L}x$ $M = -\frac{1}{2}\frac{W}{L}x^2;$ Max $M = -\frac{1}{2}WL$ at B $y = -\frac{1}{24}\frac{W}{EIL}(x^4-4L^3x+3L^4);$ Max $y = -\frac{1}{8}\frac{WL^3}{EI}$ $\theta = +\frac{1}{6}\frac{WL^2}{EI}$ at A
4. 	$R_D = +W;$ (A to B)$V = 0$: (B to C)$V = -\frac{W}{b-a}(x-L+b);$ (C to D)$V = -W$ (A to B)$M = 0;$ (B to C)$M = -\frac{1}{2}\frac{W}{b-a}(x-L+b)^2;$ (C to D)$M = -\frac{1}{2}W(2x-2L+a+b);$ Max $M = -\frac{1}{2}W(a+b)$ at D (A to B)$y = -\frac{1}{24}\frac{W}{EI}\left[4(a^2+ab+b^2)(L-x)-a^3-ab^2-a^2b-b^3\right]$ (B to C)$y = -\frac{1}{24}\frac{W}{EI}\left[6(a+b)(L-x)^2-4(L-x)^3+\frac{(L-x-a)^4}{b-a}\right];$ (C to D)$y = -\frac{1}{12}\frac{W}{EI}\left[3(a+b)(L-x)^2-2(L-x)^3\right]$ Max $y = -\frac{1}{24}\frac{W}{EI}\left[4(a^2+ab+b^2)L-a^3-ab^2-a^2b-b^3\right]$ at A; $\theta = +\frac{1}{6}\frac{W}{EI}\left(a^2+ab+b^2\right)$ (A to B)

Table 4.1.1-2 Beam Formulas (Cont'd)

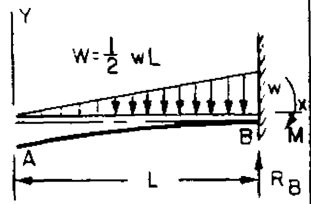
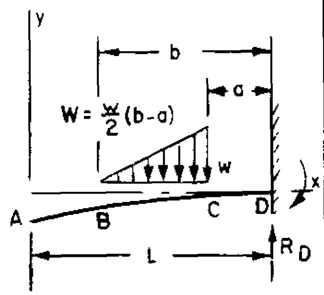
<table>
<tr><th colspan="2">Cantilever Beams</th></tr>
<tr><th>Type of loading and case number</th><th>Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope</th></tr>
<tr><td>5.

</td><td>$R_B = +W;\ V = -\frac{W}{L^2}x^2$
$M = -\frac{1}{3}\frac{W}{L^2}x^3$; Max $M = -\frac{1}{3}WL$ at B
$y = -\frac{1}{60}\frac{W}{EIL^2}(x^5-5L^4x+4L^5)$; Max $y = -\frac{1}{15}\frac{WL^3}{EI}$ at A;
$\theta = +\frac{1}{12}\frac{WL^2}{EI}$ at A</td></tr>
<tr><td>6.

</td><td>$R_D = +W$; (A to B) $V = 0$; (B to C) $V = -\frac{W(x-L+b)^2}{(b-a)^2}$; (C to D) $V = -W$
(A to B) $M = 0$; (B to C) $M = -\frac{1}{3}\frac{W(x-L+b)^3}{(b-a)^2}$;
(C to D) $M = -\frac{1}{3}W(3x-3L+b+2a)$; Max $M = -\frac{1}{3}W(b+2a)$ at D
(A to B) $y = -\frac{1}{60}\frac{W}{EI}\left[(5b^2+10ba+15a^2)(L-x) - 4a^3 - 2ab^2 - 3a^2b - b^3\right]$
(B to C) $y = -\frac{1}{60}\frac{W}{EI}\left[(20a+10b)(L-x)^2 - 10(L-x)^3 + 5\frac{(L-x-a)^4}{b-a} - \frac{(L-x-a)^5}{(b-a)^2}\right]$;
(C to D) $y = -\frac{1}{6}\frac{W}{EI}\left[(2a+b)(L-x)^2 - (L-x)^3\right]$;
Max $Y = -\frac{1}{60}\frac{W}{EI}\left[(5b^2+10ba+15a^2)L - 4a^3 - 2ab^2 - 3a^2b - b^3\right]$ at A
$\theta = +\frac{1}{12}\frac{W}{EI}(3a^2+2ab+b^2)$ (A to B)</td></tr>
</table>

Table 4.1.1-2 Beam Formulas (Cont'd)

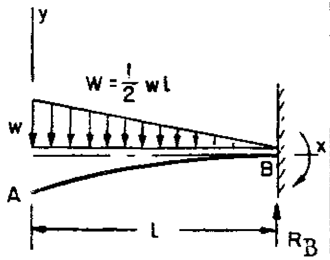
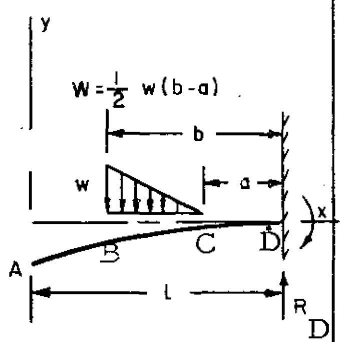
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
7. 	$R_B = +W;\ V = -W\left(\frac{2Lx-x^2}{L^2}\right)$ $M = -\frac{1}{3}\frac{W}{L^2}(3Lx^2-x^3)$; Max $M = -\frac{2}{3}WL$ at B $y = -\frac{1}{60}\frac{W}{EIL^2}(-x^5-15L^4x+5LX^4+11L^5)$; Max $Y = -\frac{11}{60}\frac{WL^3}{EI}$ at A $\theta = +\frac{1}{4}\frac{W}{EI}L^2$ at A
8. 	$R_D = +W$; (A to B) $V = 0$; (B to C) $V = -W\left[1 - \frac{(L-a-x)^2}{(b-a)^2}\right]$; (C to D) $V = -W$ (A to B) $M = 0$; (B to C) $M = -\frac{1}{3}W\left[\frac{3(x-L+b)^2}{b-a} - \frac{(x-L+b)^3}{(b-a)^2}\right]$ (C to D) $M = -\frac{1}{3}W(-3L+3x+2b+a)$; Max $M = -\frac{1}{3}W(2b+a)$ at D (A to B) $y = -\frac{1}{60}\frac{W}{EI}\left[(5a^2+10ab+15b^2)(L-x)-a^3-2a^2b-3ab^2-4b^3\right]$; (B to C) $y = -\frac{1}{60}\frac{W}{EI}\left[\frac{(L-x-a)^5}{(b-a)^2} -10(L-x)^3+(10a+20b)(L-x)^2\right]$ (C to D) $y = -\frac{1}{6}\frac{W}{EI}\left[(a+2b)(L-x)^2 -(L-x)^3\right]$ Max $Y = -\frac{1}{60}\frac{W}{EI}\left[(5a^2+10ab+15b^2)L -a^3-2a^2b-3ab^2-4b^3\right]$ at A $\theta = +\frac{1}{12}\frac{W}{EI}(a^2+2ab+3b^2)$ (A to B)

Table 4.1.1-2 Beam Formulas (Cont'd)

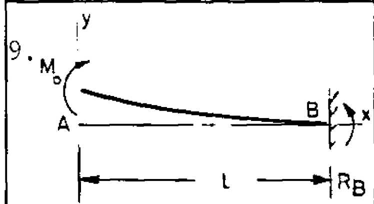
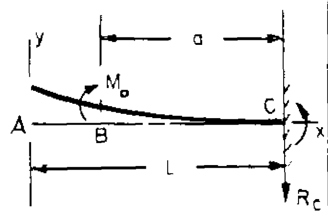
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
9. 	$R_B = 0$; $V = o$ $M = M_o$; Max $M = M_o$ (A to B) $y = \frac{1}{2}\frac{M_o}{EI}(L^2-2Lx+x^2)$; Max $y = \frac{1}{2}\frac{M_oL^2}{EI}$ at A; $\theta = -\frac{M_oL}{EI}$ at A
10. 	$R_C = 0$; $V = 0$ (A to B) $M = 0$; (B to C) $M = M_o$; Max $M = M_o$ (B to C) (A to B) $y = \frac{M_oa}{EI}\left(L - \frac{1}{2}a - x\right)$; (B to C) $y = \frac{1}{2}\frac{M_o}{EI}\left[(x-L+a)^2 - 2a(x-L+a) + a^2\right]$ Max $y = \frac{M_oa}{EI}\left(L - \frac{1}{2}a\right)$ at A; $\theta = -\frac{M_oa}{EI}$ (A to B)

Table 4.1.1-2 Beam Formulas (Cont'd)

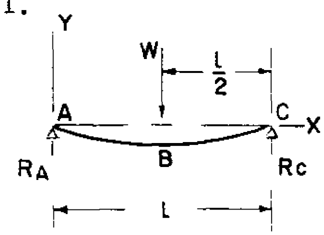
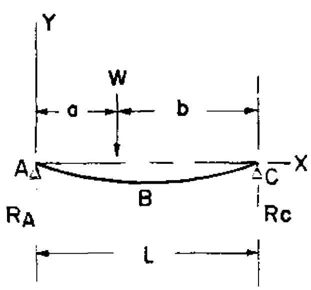
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
11. 	$R_A = \frac{1}{2} W$; $R_C = \frac{1}{2} W$; (A to B) $V = \frac{1}{2} W$; (B to C) $V = -\frac{1}{2} W$ (A to B) $M = \frac{1}{2} Wx$; (B to C) $M = \frac{1}{2} W (L-x)$; Max $M = \frac{1}{4} WL$ at B (A to B) $y = -\frac{1}{48} \frac{W}{EI} (3L^2x-4x^3)$; Max $y = -\frac{1}{48} \frac{WL^3}{EI}$ at B; $\theta = -\frac{1}{16} \frac{WL^2}{EI}$ at A; $\theta = +\frac{1}{16} \frac{WL^2}{EI}$ at C
12. 	$R_A = +\frac{Wb}{L}$; $R_C = +\frac{Wa}{L}$; (A to B) $V = +\frac{Wb}{L}$; (B to C) $V = -\frac{Wa}{L}$ (A to B) $M = +\frac{Wb}{L} x$; (B to C) $M = +\frac{Wa}{L} (L-x)$; Max $M = +\frac{Wab}{L}$ at B (A to B) $y = -\frac{Wbx}{6EIL} \left[2L(L-x) - b^2 - (L-x)^2\right]$; (B to C) $y = -\frac{Wa(L-x)}{6\ EIL} \left[2\ Lb - b^2 - (L-x)^2\right]$; Max $y = -\frac{Wab}{27EIL} (a+2b) \sqrt{3a(a+2b)}$ at $x = \sqrt{\frac{a}{3} (a+2b)}$ When $a > b$; $\theta = -\frac{1}{6} \frac{W}{EI} \left(bL - \frac{b^3}{L}\right)$ at A; $\theta = +\frac{1}{6} \frac{W}{EI} \left(2bL + \frac{b^3}{L} - 3b^2\right)$ at C

Table 4.1.1-2 Beam Formulas (Cont'd)

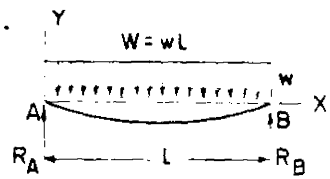
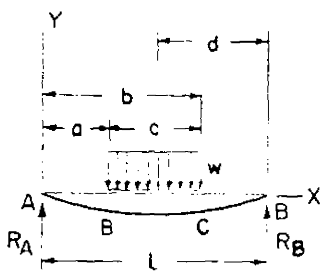
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
13. 	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; $V = +\frac{W}{2}\left(1 - \frac{2x}{L}\right)$; $M = +\frac{W}{2}\left(x - \frac{x^2}{L}\right)$ Max. $M = +\frac{WL}{8}$ at $x = \frac{L}{2}$; $y = -\frac{Wx}{24EIL}(L^3 - 2Lx^2 + x^3)$; Max. $y = -\frac{5WL^3}{384EI}$ at $x = \frac{L}{2}$; $\theta = -\frac{WL^2}{24EI}$ at A; $\theta = +\frac{WL^2}{24EI}$ at B
14.  $W = wc$ $d = L - \frac{1}{2}b - \frac{1}{2}a$	$R_A = \frac{Wd}{L}$; $R_D = \frac{W}{L}\left(a + \frac{c}{2}\right)$; (A to B) $V = R_A$; (B to C) $V = R_A - \frac{W(x-a)}{c}$ (C to D) $V = R_A - W$; (A to B) $M = R_A x$; (B to C) $M = R_A x - \frac{W(x-a)^2}{2c}$ (C to D) $M = R_A x - W\left(x - \frac{a}{2} - \frac{b}{2}\right)$; Max. $M = \frac{Wd}{L}\left(a + \frac{cd}{2L}\right)$ at $x = a + \frac{cd}{L}$ (A to B) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right]\right\}$ (B to C) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right] - \frac{2W(x-a)^4}{c}\right\}$ (C to D) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L}\right] - 8W\left(x - \frac{a}{2} - \frac{b}{2}\right)^3 + W(2bc^2 - c^3)\right\}$

Table 4.1.1-2 Beam Formulas (Cont'd)

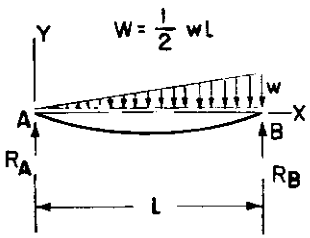
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
14. (Cont'd)	$\theta = \frac{1}{48EI}\left[-8R_A L^2 + W\left(\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right)\right]$ at A $\theta = \frac{1}{48EI}\left[16R_A L^2 - W\left(24d^2 - \frac{8d^3}{L} + \frac{2bc^2}{L} - \frac{c^3}{L}\right)\right]$ at B
15. 	$R_A = \frac{W}{3}$; $R_B = \frac{2W}{3}$; $V = W\left(\frac{1}{3} - \frac{x^2}{L^2}\right)$; $M = \frac{W}{3}\left(x - \frac{x^3}{L^2}\right)$; Max. M = 0.128 WL at $x = \frac{L\sqrt{3}}{3}$; $y = \frac{-Wx(3x^4 - 10L^2x^2 + 7L^4)}{180EIL^2}$ Max. $y = -\frac{0.01304\ WL^3}{EI}$ at x = 0.519 L $\theta = -\frac{7WL^2}{180EI}$ at A ; $\theta = \frac{8WL^2}{180EI}$ at B

Table 4.1.1-2 Beam Formulas (Cont'd)

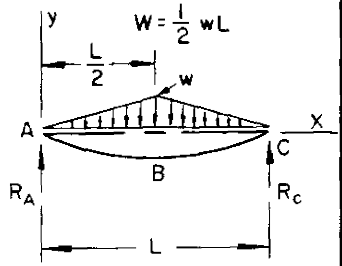
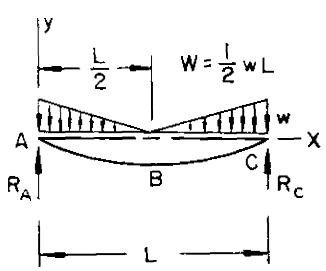
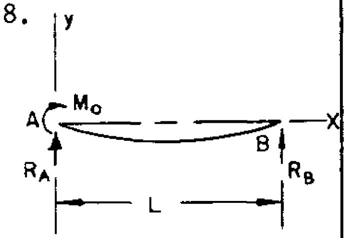
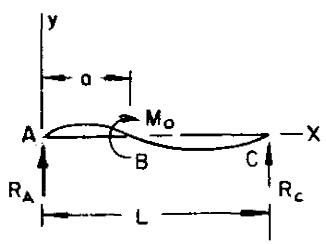
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
16. 	$R_A = \frac{W}{2}$; $R_C = \frac{W}{2}$; (A to B) $V = \frac{W}{2}\left(1 - \frac{4x^2}{L^2}\right)$; (B to C) $V = -\frac{W}{2}\left[1 - \frac{4(L-x)^2}{L^2}\right]$; (A to B) $M = \frac{W}{6}\left(3x - \frac{4x^3}{L^2}\right)$ (B to C) $M = \frac{W}{6}\left[3(L-x) - \frac{4(L-x)^3}{L^2}\right]$; Max. $M = \frac{WL}{6}$ at B (A to B) $y = \frac{Wx}{6EIL^2}\left(\frac{L^2x^2}{2} - \frac{x^4}{5} - \frac{5L^4}{16}\right)$; Max. $y = -\frac{WL^3}{60EI}$ at B $\theta = -\frac{5WL^2}{96EI}$ at A : $\theta = +\frac{5WL^2}{96EI}$ at C
17. 	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; (A to B) $V = \frac{W}{2}\left(\frac{L-2x}{L}\right)^2$ (B to C) $V = -\frac{W}{2}\left(\frac{2x-L}{L}\right)^2$; (A to B) $M = \frac{W}{2}\left(x - \frac{2x^2}{L} + \frac{4x^3}{3L^2}\right)$ (B to C) $M = \frac{W}{2}\left[(L-x) - \frac{2(L-x)^2}{L} + \frac{4(L-x)^3}{3L^2}\right]$; Max. $M = \frac{WL}{12}$ at B (A to B) $y = \frac{W}{12EI}\left(x^3 - \frac{x^4}{L} + \frac{2x^5}{5L^2} - \frac{3L^2x}{8}\right)$; Max. $y = -\frac{3WL^3}{320EI}$ at B $\theta = -\frac{WL^2}{32EI}$ at A; $\theta = \frac{WL^2}{32EI}$ at C

Table 4.1.1-2 Beam Formulas (Cont'd)

<table>
<tr><th colspan="2">Simply Supported Beams</th></tr>
<tr><th>Type of loading and case number</th><th>Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope</th></tr>
<tr><td>18.
</td><td>$R_A = -\frac{M_o}{L}$; $R_B = \frac{M_o}{L}$; $V = R_A$; $M = M_o + R_A x$; Max. $M = M_o$ at A

$y = \frac{M_o}{6EI}\left(3x^2 - \frac{x^3}{L} - 2Lx\right)$; Max. $y = -0.0642\frac{M_oL^2}{EI}$ at $x = 0.422L$

$\theta = -\frac{M_oL}{3EI}$ at A; $\theta = \frac{M_oL}{6EI}$ at B</td></tr>
<tr><td>19.
</td><td>$R_A = -\frac{M_o}{L}$; $R_C = \frac{M_o}{L}$; (A to C) $V = +R_A$; (A to B) $M = +R_A x$

(B to C) $M = +R_A x + M_o$; Max. $(-M) = +R_A a$ just left of B

Max. $(+M) = +R_A a + M_o$ just right of B

(A to B) $y = \frac{M_o}{6EI}\left[\left(6a - \frac{3a^2}{L} - 2L\right)x - \frac{x^3}{L}\right]$

(B to C) $y = \frac{M_o}{6EI}\left[3a^2 + 3x^2 - \frac{x^3}{L} - \left(2L + \frac{3a^2}{L}\right)x\right]$

$\theta = -\frac{M_o}{6EI}\left(2L - 6a + \frac{3a^2}{L}\right)$ at A; $\theta = \frac{M_o}{6EI}\left(L - \frac{3a^2}{L}\right)$ at C

$\theta = \frac{M_o}{EI}\left(a - \frac{a^2}{L} - \frac{L}{3}\right)$ at B</td></tr>
</table>

Table 4.1.1-2 Beam Formulas (Cont'd)

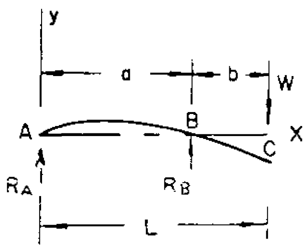
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
20.  	$R_A = \frac{-Wb}{a}$; $R_B = \frac{WL}{a}$; (A to B) $V = +R_A$; (B to C) $V = +W$; (A to B) $M = +R_A x$; (B to C) $M = +R_A a + W(x-a)$; Max. $M = +R_A a$ at B (A to B) $y = \frac{-Wbx}{6aEI}(x^2 - a^2)$; (B to C) $y = \frac{-W}{6EI}\left[(L-x)^3 - b(L-x)(2L-b) + 2b^2L\right]$ Max. $y = \frac{-Wb^2L}{3EI}$ at C; $\theta = \frac{Wab}{6EI}$ at A; $\theta = -\frac{Wab}{3EI}$ at B $\theta = -\frac{Wb}{6EI}(2L+b)$ at C

Table 4.1.1-2 Beam Formulas (Cont'd)

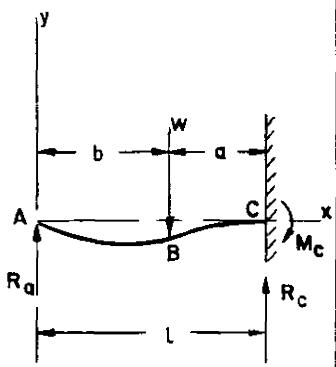
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
21.  	$R_A = \frac{W}{2}\left(\frac{3a^2L-a^3}{L^3}\right)$; $R_C = W - R_A$; $M_c = \frac{W}{2}\left(\frac{a^3+2aL^2-3a^2L}{L^2}\right)$ (A to B) $V = R_A$; (B to C) $V = R_A - W$; (A to B) $M = R_Ax$ (B to C) $M = R_Ax - W(x-L+a)$; Max. $(+M) = R_A(L-a)$ at B max. possible value = 0.174 WL when a = 0.634 L Max. $(-M) = -M_c$ at C max. possible value = -0.1927 WL when a = 0.4227 L (A to B) $y = \frac{1}{6EI}\left[R_A(x^3-3L^2x) + 3Wa^2x\right]$ (B to C) $y = \frac{1}{6EI}\left\{R_A(x^3-3L^2x) + W\left[3a^2x - (x-b)^3\right]\right\}$ If $a < 0.586$ L, max. y is between A and B at $x = L\sqrt{1 - \frac{2L}{3L-a}}$ If $a > 0.586$ L, max. y is at $x = \frac{L(L^2+b^2)}{3L^2-b^2}$ If a = 0.586 L, max. y is at B and $= -0.0098\,\frac{WL^3}{EI}$, max. possible deflection $\theta = \frac{W}{4EI}\left(\frac{a^3}{L} - a^2\right)$ at A

Table 4.1.1-2 Beam Formulas (Cont'd)

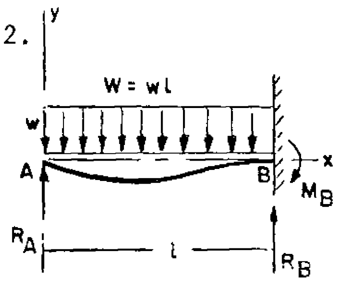
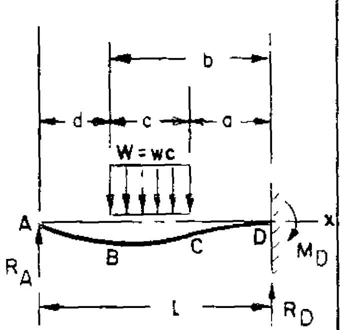
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
22. 	$R_A = \frac{3W}{8}$; $R_B = \frac{5W}{8}$; $M_B = \frac{WL}{8}$; $V = W\left(\frac{3}{8} - \frac{x}{L}\right)$; $M = W\left(\frac{3x}{8} - \frac{x^2}{2L}\right)$ Max. (+M) $= \frac{9WL}{128}$ at $x = \frac{3L}{8}$; Max. (-M) $= -\frac{WL}{8}$ at B $y = \frac{W}{48EIL}(3Lx^3 - 2x^4 - L^3x)$; Max. $y = -0.0054\frac{WL^3}{EI}$ at $x = 0.4215\ L$ $\theta = -\frac{WL^2}{48EI}$ at A
23. 	$R_A = \frac{W}{8L^3}\left[4L(a^2+ab+b^2) - a^3 - ab^2 - a^2b - b^3\right]$; $R_D = W - R_A$; $M_D = -R_AL + \frac{W}{2}(a+b)$; (A to B) $V = R_A$; (B to C) $V = R_A - W\left(\frac{x-d}{c}\right)$ (C to D) $V = R_A - W$; (A to B) $M = R_Ax$; (B to C) $M = R_Ax - \frac{W(x-d)^2}{2c}$ (C to D) $M = R_Ax - W\left(x - d - \frac{c}{2}\right)$; Max. (+M) $= R_A\left(d + \frac{R_Ac}{2W}\right)$ at $x = \left(d + \frac{R_Ac}{W}\right)$; Max. (-M) $= -M_D$ (A to B) $y = \frac{1}{EI}\left[R_A\left(\frac{x^3}{6} - \frac{L^2x}{2}\right) + Wx\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right)\right]$ (B to C) $y = \frac{1}{EI}\left[R_A\left(\frac{x^3}{6} - \frac{L^2x}{2}\right) + Wx\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right) - \frac{W(x-d)^4}{24c}\right]$

Table 4.1.1-2 Beam Formulas (Cont'd)

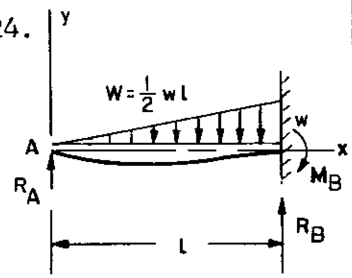
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
23. (Cont'd)	(C to D) $y = \frac{1}{EI}\left\{R_A\left(\frac{x^3}{6} - \frac{L^2x}{2} + \frac{L^3}{3}\right) + W\left[\frac{1}{6}\left(a + \frac{c}{2}\right)^3 - \frac{1}{2}\left(a + \frac{c}{2}\right)^2 L - \frac{1}{6}\left(x - d - \frac{c}{2}\right)^3 + \frac{1}{2}\left(a + \frac{c}{2}\right)^2 x\right]\right\}$ $\theta = -\frac{1}{EI}\left[\frac{R_A L^2}{2} - W\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right)\right]$ at A
24. 	$R_A = \frac{W}{5}$; $R_B = \frac{4W}{5}$; $M_B = \frac{2WL}{15}$; $V = W\left(\frac{1}{5} - \frac{x^2}{L^2}\right)$; $M = W\left(\frac{x}{5} - \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.06 WL at x = 0.4474 L; Max. (-M) = $-M_B$ $y = \frac{W}{60EIL}\left(2Lx^3 - L^3x - \frac{x^5}{L}\right)$; Max. y = $-0.00477\frac{WL^3}{EI}$ at $x = L\sqrt{\frac{1}{5}}$ $\theta = -\frac{WL^2}{60EI}$ at A

Table 4.1.1-2 Beam Formulas (Cont'd)

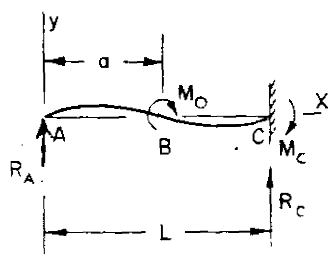
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
25.  	$R_A = -\frac{3M_o}{2L}\left(\frac{L^2-a^2}{L^2}\right)$; $R_C = \frac{3M_o}{2L}\left(\frac{L^2-a^2}{L^2}\right)$; $M_c = \frac{M_o}{2}\left(1 - \frac{3a^2}{L^2}\right)$ (A to B) $V = +R_A$; (B to C) $V = +R_A$; (A to B) $M = +R_A x$ (B to C) $M = +R_A x + M_o$; Max. $(+M) = M_o\left[1 - \frac{3a(L^2-a^2)}{2L^3}\right]$ just to the right of B Max. $(-M) = -M_c$ at C when $a < 0.275\ L$ Max. $(-M) = R_A a$ to the left of B when $a > 0.275\ L$ (A to B) $y = \frac{M_o}{EI}\left[\frac{L^2-a^2}{4L^3}(3L^2x-x^3) - (L-a)x\right]$ (B to C) $y = \frac{M_o}{EI}\left[\frac{L^2-a^2}{4L^3}(3L^2x-x^3) - Lx + \frac{(x^2+a^2)}{2}\right]$ $\theta = \frac{M_o}{EI}\left(a - \frac{L}{4} - \frac{3a^2}{4L}\right)$ at A

Table 4.1.1-2 Beam Formulas (Cont'd)

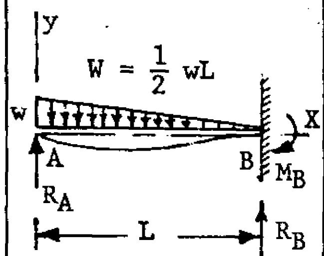
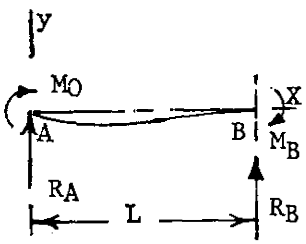
Type of loading and case number	Statically Indeterminate Cases: Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
26. 	$R_A = \frac{11W}{20}$; $R_B = \frac{9W}{20}$; $M_B = \frac{7WL}{60}$; $V = W\left(\frac{11}{20} - \frac{2x}{L} + \frac{x^2}{L^2}\right)$; $M = W\left(\frac{11x}{20} - \frac{x^2}{L} + \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.0846 WL at x = 0.329 L; Max. (-M) = $-\frac{7WL}{60}$ at B $y = \frac{W}{120EIL}\left(11Lx^3 - 3L^3x - 10x^4 + \frac{2x^5}{L}\right)$ Max. y = $-0.00609\ \frac{WL^3}{EI}$ at x = 0.402 L; $= \frac{WL^2}{40EI}$ at A
27. 	$R_A = -\frac{3M_O}{2L}$; $R_B = \frac{3M_O}{2L}$; $M_B = \frac{M_O}{2}$; $V = -\frac{3M_O}{2L}$; $M = \frac{M_O}{2}\left(2 - \frac{3x}{L}\right)$ Max. (+M) = M_O at A; Max. (-M) = $-\frac{M_O}{2}$ at B $y = \frac{M_O}{4EI}\left(2x^2 - \frac{x^3}{L} - Lx\right)$; Max. y = $-\frac{M_OL^2}{27EI}$ at x = $\frac{L}{3}$ $\theta = -\frac{M_OL}{4EI}$ at A

Table 4.1.1-2 Beam Formulas (Cont'd)

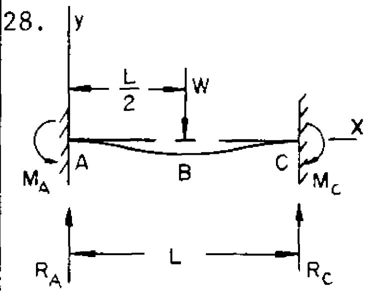
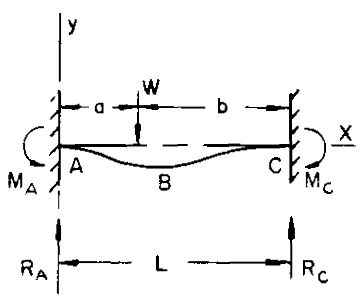
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
28. 	$R_A = \frac{W}{2}$; $R_C = \frac{W}{2}$; $M_A = \frac{WL}{8}$; $M_c = \frac{WL}{8}$; (A to B) $V = \frac{W}{2}$; (B to C) $V = -\frac{W}{2}$ (A to B) $M = \frac{W(4x-L)}{8}$; (B to C) $M = \frac{W(3L-4x)}{8}$; Max. $(+M) = \frac{WL}{8}$ at B Max. $(-M) = -\frac{WL}{8}$ at A and C; (A to B) $y = -\frac{W}{48EI}(3Lx^2-4x^3)$ Max. $y = -\frac{WL^3}{192EI}$ at B
29. 	$R_A = \frac{Wb^2}{L^3}(3a+b)$; $R_C = \frac{Wa^2}{L^3}(3b+a)$; $M_a = \frac{Wab^2}{L^2}$; $M_c = \frac{Wba^2}{L^2}$ (A to B) $V = R_A$; (B to C) $V = R_A - W$; (A to B) $M = -\frac{Wab^2}{L^2} + R_A x$ (B to C) $M = -\frac{Wab^2}{L^2} + R_A x - W(x-a)$; Max. $(+M) = -\frac{Wab^2}{L^2} + R_A a$ at B, max. value $= \frac{WL}{8}$ when $a = \frac{L}{2}$ Max. $(-M) = -M_A$ when $a < b$, max. possible value $= -0.1481$ WL when $a = \frac{L}{3}$ Max. $(-M) = -M_c$ when $a > b$, max. possible value $= -0.1481$ WL when $a = \frac{2L}{3}$

Table 4.1.1-2 Beam Formulas (Cont'd)

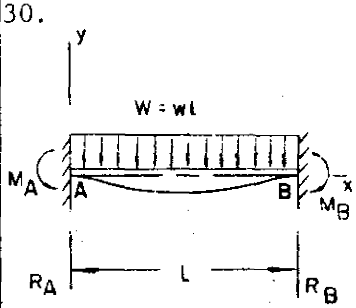
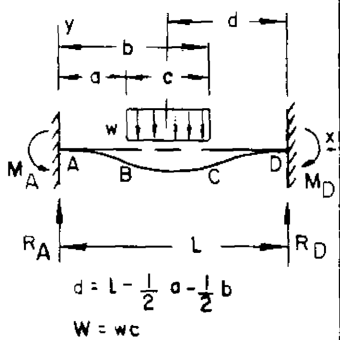
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
29. (Cont'd)	(A to B) $y = \frac{Wb^2x^2}{6EIL^3}(3ax + bx - 3aL)$ (B to C) $y = \frac{Wa^2(L-x)^2}{6EIL^3}\left[(3b+a)(L-x) - 3bL\right]$ Max. $y = -\frac{2Wa^3b^2}{3EI(3a+b)^2}$ at $x = \frac{2aL}{3a+b}$ if $a > b$ Max. $y = -\frac{2Wa^2b^3}{3EI(3b+a)^2}$ at $x = L - \frac{2bL}{3b+a}$ if $a < b$
30. 	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; $M_A = \frac{WL}{12}$; $M_B = \frac{WL}{12}$; $V = \frac{W}{2}\left(1 - \frac{2x}{L}\right)$ $M = \frac{W}{2}\left(x - \frac{x^2}{L} - \frac{L}{6}\right)$; Max. $(+M) = \frac{WL}{24}$ at $x = \frac{L}{2}$ Max. $(-M) = -\frac{WL}{12}$ at A and B; $y = \frac{Wx^2}{24EIL}(2Lx - L^2 - x^2)$ Max. $y = \frac{-WL^3}{384EI}$ at $x = \frac{L}{2}$

Table 4.1.1-2 Beam Formulas (Cont'd)

Statically Indeterminate Cases

Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope

31.



$$R_A = \frac{W}{4L^2}\left(12d^2 - \frac{8d^3}{L} + \frac{2bc^2}{L} - \frac{c^3}{L} - c^2\right);\ R_D = W - R_A$$

$$M_A = -\frac{W}{24L}\left(\frac{24d^3}{L} - \frac{6bc^2}{L} + \frac{3c^3}{L} + 4c^2 - 24d^2\right)$$

$$M_D = \frac{W}{24L}\left(\frac{24d^3}{L} - \frac{6bc^2}{L} + \frac{3c^3}{L} + 2c^2 - 48d^2 + 24dL\right);\ (A\ to\ B)\ V = R_A$$

$$(B\ to\ C)\ V = R_A - W\left(\frac{x-a}{c}\right);\ (C\ to\ D)\ V = R_A - W;\ (A\ to\ B)\ M = -M_A + R_A x$$

$$(B\ to\ C)\ M = -M_A + R_A x - W\frac{(x-a)^2}{2c};\ (C\ to\ D)\ M = -M_A + R_A x - W(x-L+d)$$

Max. (+M) is between B and C at $x = a + \frac{R_A c}{W}$;

Max. (-M) = $-M_A$ when $a < (L-b)$; Max. (-M) = $-M_D$ when $a > (L-b)$

$$(A\ to\ B)\ y = \frac{1}{6EI}\left(R_A x^3 - 3M_A x^2\right)$$

$$(B\ to\ C)\ y = \frac{1}{6EI}\left(R_A x^3 - 3M_A x^2 - \frac{W(x-a)^4}{4c}\right)$$

$$(C\ to\ D)\ y = \frac{1}{6EI}\left[R_D(L-x)^3 - 3M_D(L-x)^2\right]$$

Table 4.1.1-2 Beam Formulas (Cont'd)

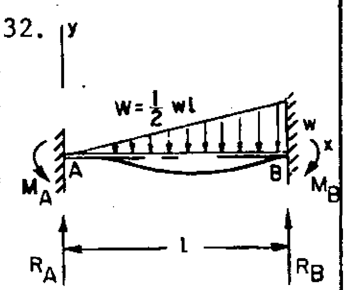
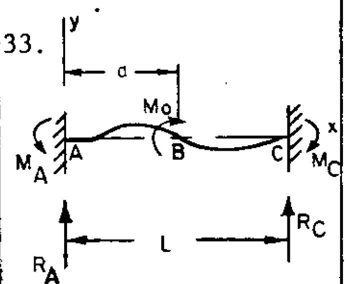
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
32. 	$R_A = \frac{3W}{10}$; $R_B = \frac{7W}{10}$; $M_A = \frac{WL}{15}$; $M_B = \frac{WL}{10}$; $V = W\left(\frac{3}{10} - \frac{x^2}{L^2}\right)$ $M = W\left(\frac{3x}{10} - \frac{L}{15} - \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.043 WL at x = 0.548 L; Max. (-M) = $-\frac{WL}{10}$ at B $y = \frac{W}{60EI}\left(3x^3 - 2Lx^2 - \frac{x^5}{L^2}\right)$; Max. $y = -0.002617 \frac{WL^3}{EI}$ at x = 0.525 L
33. 	$R_A = -\frac{6M_o}{L^3}(aL-a^2)$; $R_C = \frac{6M_o}{L^3}(aL-a^2)$; $M_A = -\frac{M_o}{L^2}(4La-3a^2-L^2)$ $M_C = \frac{M_o}{L^2}(2La-3a^2)$; $V = +R_A$; (A to B) $M = -M_A + R_A x$ (B to C) $M = -M_A + R_A x + M_o$ Max. (+M) $= M_o\left(\frac{4a}{L} - \frac{9a^2}{L^2} + \frac{6a^3}{L^3}\right)$ just to the right of B Max. (-M) $= M_o\left(\frac{4a}{L} - \frac{9a^2}{L^2} + \frac{6a^3}{L^3} - 1\right)$ just to the left of B (A to B) $y = \frac{-1}{6EI}(3M_A x^2 - R_A x^3)$

Table 4.1.1-2 Beam Formulas (Cont'd)

	Statically Indeterminate Cases
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
33. (Cont'd)	(B to C) $y = \frac{1}{6EI}\left[(M_o - M_A)(3x^2 - 6Lx + 3L^2) - R_A(3L^2x - x^3 - 2L^3)\right]$ Max (-y) at $x = L - \frac{2M_c}{R_c}$ if $a < \frac{2L}{3}$ Max (+y) at $x = \frac{2M_A}{R_A}$ if $a > \frac{L}{3}$

B4.1.2 Stress Analysis

The maximum bending stress is:

$$f = \frac{Mc}{I} \qquad (1)$$

The limitations are:

a) The loads on the beam must be static loads.
b) The value of f is the result of external forces only.
c) The beam acts as a unit with bending as the dominant action.
d) The initial curvature of the member must be relatively small. (Radius of curvature at least ten times the depth.)
e) Plane cross sections remain plane.
f) The material follows Hooke's law.

If the calculated stress does exceed the proportional limit, a suitable reduced modulus must be used.

The maximum shearing stress in a beam in combined bending and shear is:

$$f_s = K \frac{V}{A} \qquad (2)$$

where (K) is the ratio of the maximum shearing stress on the cross section to the average shearing stress. The maximum shearing stress is often expressed as:

$$f_s = \frac{VQ}{It} \qquad (3)$$

Where $Q = \int_{Area} y\,dA$ (First moment about the neutral axis of the area between the neutral axis and the extreme outer fiber.)

B 4.1.3 Variable Cross Section

The following formulas and figures present a method of analyzing beams with uniformly tapering cross sections. Figure B 4.1.3-1 shows a tapered cantilever beam consisting of two concentrated flange areas joined by a vertical web which resists no bending. The vertical components of the loads in the flanges, P tan α_1, and P tan α_2, resist some of the external force V. Letting V_f equal the force resisted by the flanges and V_w the force resisted by the webs, then:

$$V = V_f + V_w \qquad (1)$$

$$V_f = P\ (\tan \alpha_1, + \tan \alpha_2) \qquad (2)$$

From Fig. B 4.1.3-1, $\tan \alpha_1 = \frac{h_1}{c}$, $\tan \alpha_2 = \frac{h_2}{c}$, and $\tan \alpha_1 + \tan \alpha_2 = \frac{h_1 + h_2}{c} = \frac{h}{c}$. From this $V_f = P \frac{h}{c}$, and since $P = V \frac{b}{h}$,

then

$$V_f = V \frac{b}{c} \qquad (3)$$

The load in the web is $V \frac{a}{c}$, so by writing a, b, and c, in terms of h_o and h, we have

$$V_w = V \frac{h_o}{h} \qquad (4)$$

$$V_f = V \frac{(h - h_o)}{h} \qquad (5)$$

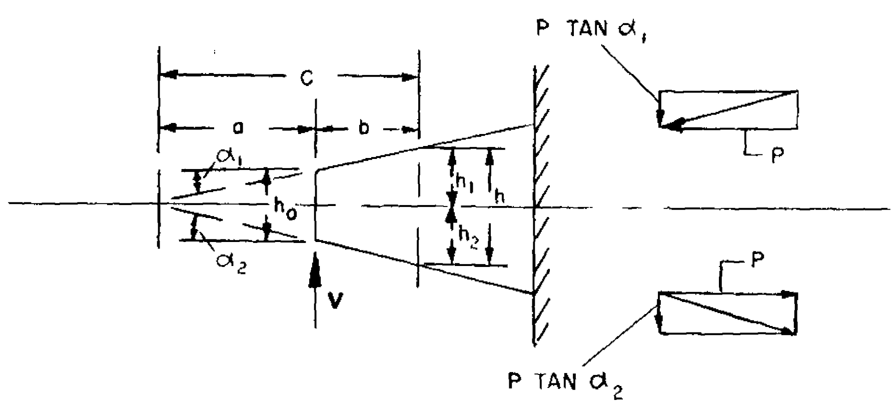


Fig. B 4.1.3-1

B 4.1.4 Symmetrical Beams of Two Different Materials

The analysis of symmetrical beams of two or more materials within the elastic range may be analyzed by transforming the section into an equivalent beam of one material. The usual elastic flexure formula then applies.

The transformation is accomplished by changing the dimension perpendicular to the axis of symmetry of the various materials in the ratio of their elastic moduli. Examples to illustrate the method for various conditions follows.

Example 1. Consider a beam made of two materials whose cross section is shown in Fig. B 4.1.4-1a. Assuming $n = Ea/Es$, the transformation in terms of material (S) (Fig. B 4.1.4-1b) is then $b_1 = nb$. The maximum stress in member (S) is then $f_{(s)} \max = \frac{Mh_1}{I}$ and the maximum stress in member (A) is $f_{(a)} \max = \frac{-Mh_2}{nI}$. It is noted that the section could have been transformed in terms of member (A) (Fig. B 4.1.4-1c) giving the same results for the maximum fiber stresses.

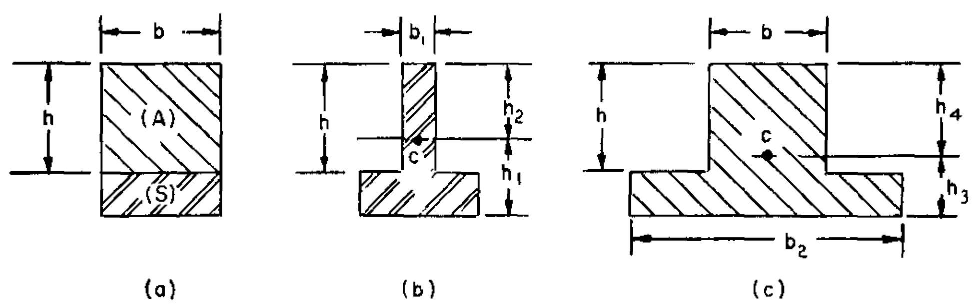


Fig. B 4.1.4-1

Example 2. Reinforced-Concrete Beams. It is the established practice in calculating bending stresses in reinforced-concrete beams to assume that concrete can withstand only compressive stress. The steel or other reinforcing member then is transformed into an equivalent area as shown in Fig. B 4.1.4-2b. The distribution of internal forces for a beam (Fig. B 4.1.4-2a) over any cross section ab is shown in Fig. B 4.1.4-2c.

B 4.1.4 Symmetrical Beams of Two Different Materials (Cont'd)

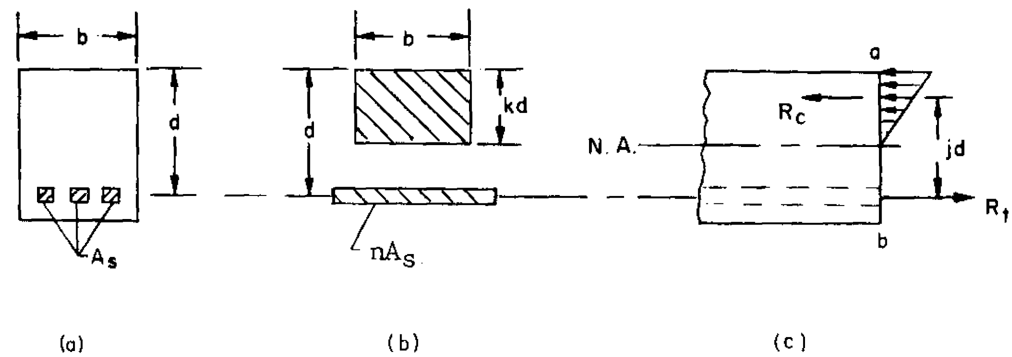


Fig. B 4.1.4-2

To satisfy the equation of equilibrium, the internal moment must equal the external moment. The mathematical statement is:

$$\underbrace{b\,(kd)}_{\text{Concrete area}}\ \underbrace{\left(\frac{kd}{2}\right)}_{\text{arm}} = \underbrace{nAs}_{\text{Transformed steel area}}\ \underbrace{(d-kd)}_{\text{arm}} \quad \cdots\cdots \quad (1)$$

From which

$$kd = \frac{nAs}{b}\left(\sqrt{1+\frac{2bd}{nAs}} - 1\right) \quad \cdots\cdots \quad (2)$$

where $n = \dfrac{E\ steel}{E\ concrete}$

The stress in the concrete f_c and the stress in the steel f_{st} is

$$f_c = \frac{M\,(kd)}{I} \quad \cdots\cdots \quad (3)$$

$$f_{st} = \frac{nM(d-kd)}{I} \quad \cdots\cdots \quad (4)$$

where

I = Moment of inertia of the transformed section.

B 4.1.4 Symmetrical Beams of Two Different Materials (Cont'd)

Alternate solution: After kd is determined, instead of computing I, a procedure evident from Fig. B 4.1.4-2c may be used. The resultant force developed by the stresses acting in a "hydrostatic" manner on the compression side of the beam must be located $\frac{kd}{3}$ below the top of the beam. Moveover, if b is the width of the beam, this resultant force $R_c = \frac{b}{2}(kd)\ f_c$ max, (average stress times area). The resultant tensile force R_t acts at the center of the steel and is equal to $A_s f_{st}$, where A_s is the cross-sectional area of the steel. Then if jd is the distance between R_c and R_t, and since $R_c = R_t$, the applied moment M is resisted by a couple equal to $R_c jd$ or $R_t jd$.

$$jd = d - \frac{1}{3} kd \qquad (5)$$

The stress in the steel and concrete is

$$f_s = \frac{M}{A_s\ jd} \qquad (6)$$

$$f_c = - \frac{2M}{b(kd)\ (jd)} \qquad (7)$$

B 4.2.0 Continuous Beams

B 4.2.1 Castigliano's Theorem

Castigliano's Theorem is useful in the solution of problems involving continuous beams with only one or two redundant supports. The theorem can be written as

$$\delta_Q = \frac{\partial U}{\partial Q} = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial Q} ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\delta_Q = \frac{\partial Us}{\partial Q} = \frac{1}{GA} \int_0^L V \frac{\partial V}{\partial Q} ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\theta_a = \frac{\partial U}{\partial M_a} = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial M_a} ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where

δ_Q is the deflection at the load Q in the direction of Q.

Q may be a real or fictitious load.

Q_a is the slope at the moment M_a in the direction of M_a.

M_a may be a real or fictitious moment.

U is the strain energy of the beam.

M is the bending moment due to all loads.

V is the vertical shearing forces due to all loads.

A is the cross-sectional area of the beam.

E is the modulus of elasticity.

I is the moment of inertia.

B 4.2.2 Unit Load or Dummy Load Method

The unit load or dummy load method may be used to determine deflection at elastic or inelastic members. Deflection of inelastic members by this method is given in section B 4.5.0. The theorem as applied to elastic beams is written in integral form as

$$\delta = \int_o^L \frac{Mm}{EI}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\theta = \int_o^L \frac{Mm'}{EI}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Where (δ) is the deflection at the unit load and (θ) is the rotation at the unit moment. The Moment (M) is the bending moment at any section caused by the actual loads. (m) is the bending moment at any section of the beam caused by a dummy load of unity acting at the point whose deflection is to be found and in the direction of the desired deflection. The bending moment (m') is the bending moment at any section of the beam caused by a dummy couple of unity applied at the section where the change in slope is desired. It is noted that although (m') may be thought of as a bending moment, it is evident from the expression $m' = \frac{\partial M}{\partial M_a}$ that it is actually dimensionless.

B 4.2.2 Unit Load or Dummy Load Method (Cont'd)

Illustrative Problem: Find the elastic vertical deflection of the point A (Fig. B 4.2.2-1a) of the simply supported beam subjected to two concentrated loads.

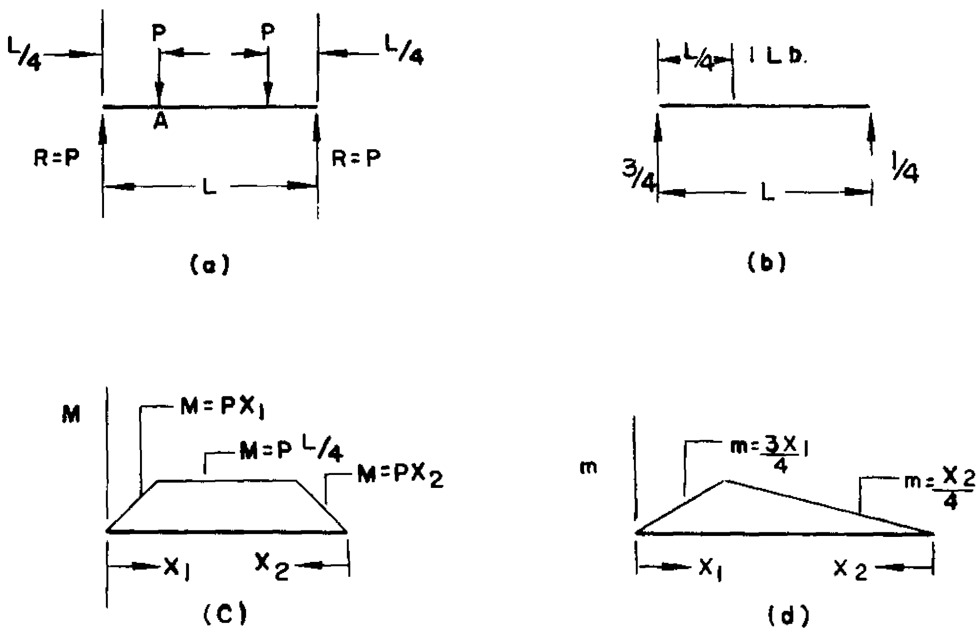


Fig. B 4.2.2-1

Solution: The actual loading is shown in Fig. B 4.2.2-1a and the dummy loading is shown on Fig. B 4.2.2-1b. The moment for the actual loading is shown on Fig. B 4.2.2-1c and the corresponding moment diagram for the dummy loading is shown on Fig. B 4.2.2-1d.

The deflection by use of equation (1) noting that x_1 starts at the left and x_2 starts at the right is

$$\delta = \int_0^L \frac{Mm}{EI}\, dx = \int_0^{\frac{L}{4}} \frac{Px_1}{EI}\left(\frac{3x_1}{4}\right) dx_1 + \int_0^{\frac{L}{4}} \frac{Px_2}{EI}\left(\frac{x_2}{4}\right) dx_2$$

$$+ \int_{\frac{L}{4}}^{\frac{3L}{4}} \frac{PL}{4EI}\left(\frac{x_2}{4}\right) dx_2 = \frac{PL^3}{48EI}$$

B 4.2.3 The Two-Moment Equation

The two-moment equation may be used to determine the bending moment at one section of the beam when the shear and bending moment at another section and the loads applied to the beam between the two sections are known. The expressions for the moment and shear corresponding to Fig. B 4.2.3-1 is

$$M_2 = M_1 + V_1 d + Fx \quad \text{(1)}$$

$$V_2 = V_1 + F \quad \text{(2)}$$

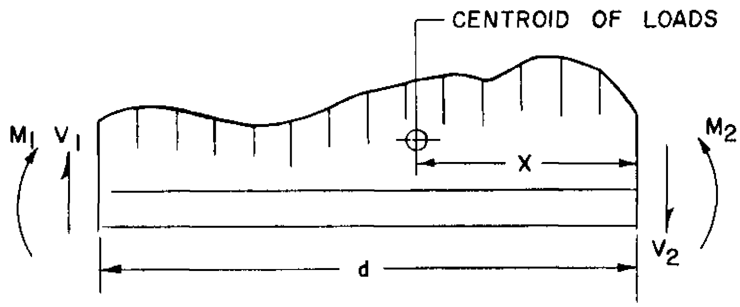


Fig. B 4.2.3-1

The two-moment equation is particularly useful in determining the curve of bending moments and shears in the case of a cantilever beam subjected to distributed loads, such as shown in Fig. B 4.2.3-2. The calculations may be done in tabular form as in Table B 4.2.3-1.

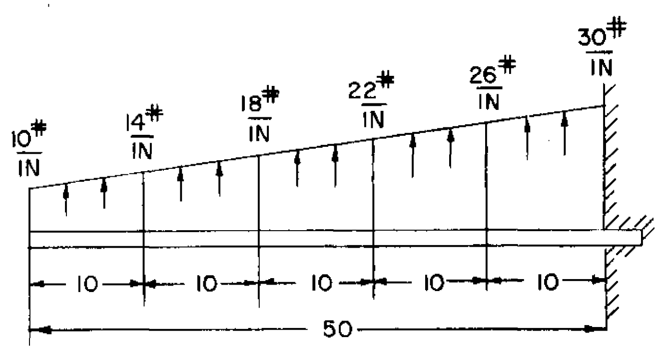


Fig. B 4.2.3-2

B 4.2.3 The Two-Moment Equation (Cont'd)

Station (inches from end)	W = Load lb/in	Shear $V=(W_1+W_2)\left(\frac{\Delta x}{2}\right)$		* Bending Moment $M=(2W_1+W_2)\left(\frac{\Delta x^2}{6}\right)+V_1(\Delta x)$	
0	10		0		0
		(10+14)10/2=	120	$(2\cdot10+14)10^2/6$ =	567
10	14		120		567
		(14+18)10/2=	160	$(2\cdot14+18)10^2/6$ = $120\cdot10$ =	767 1200
20	18		280		2534
		(18+22)10/2=	200	$(2\cdot18+22)10^2/6$ = $280\cdot10$ =	976 2800
30	22		480		6301
		(22+26)10/2=	240	$(2\cdot22+26)10^2/6$ = $480\cdot10$ =	1167 4800
40	26		720		12,268
		(26+30)10/2=	280	$(2\cdot26+30)10^2/6$ $720\cdot10$ =	1367 7200
50	30		1000		20,835

Table B 4.2.3-1

*The increment of bending moment between stations may be calculated from the relation:

$$M = (\Delta V)\text{ (Dist. from centroid of trapezoid to inboard station)}$$
$$+ V_1\ (\Delta x)$$

B 4.2.4 The Three-Moment Equation

The three-moment equation is useful in the solution of problems involving continuous beams with relatively few redundant supports. The equation is:

$$\frac{M_a L_1}{I_1} + \frac{2M_b L_1}{I_1} + \frac{2M_b L_2}{I_2} + \frac{M_c L_2}{I_2} =$$

$$K_1 + K_2 + \frac{6E}{L_1}(Y_a - Y_b) + \frac{6E}{L_2}(Y_c - Y_b) \quad \cdots\cdots \quad (1)$$

Where (K_1) and (K_2) are functions of loading on span (L_1) and span (L_2), respectively.

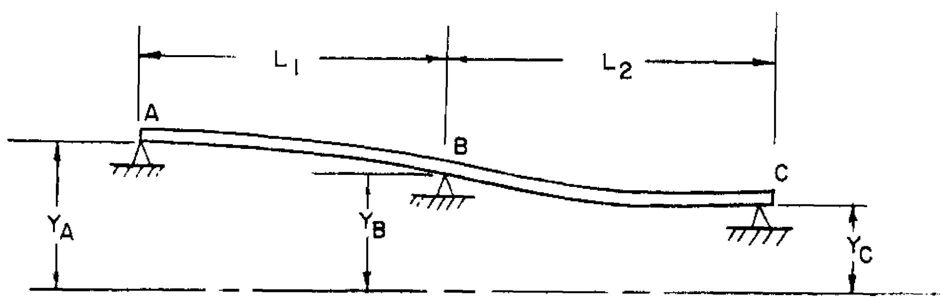


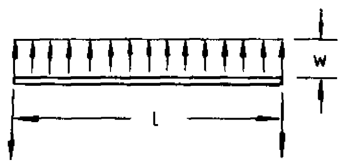
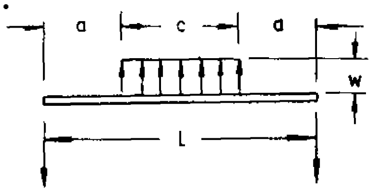
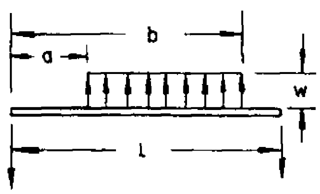
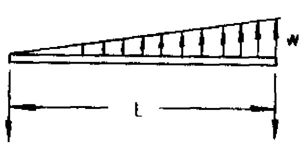
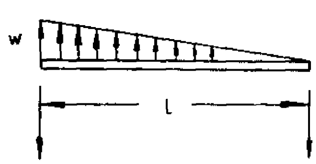
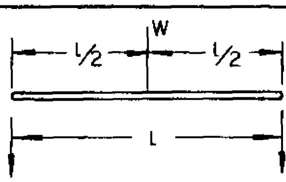
Fig. B 4.2.4-1

One equation must be written for each intermediate support. The system of simultaneous equations is then solved for the moments at the intermediate supports.

Values at (K_1) and (K_2) for various types of loading are tabulated in Table B 4.2.4-1.

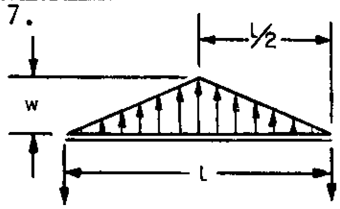
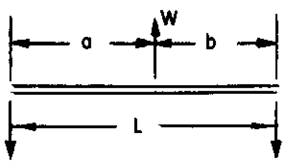
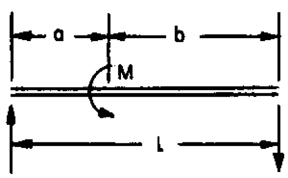
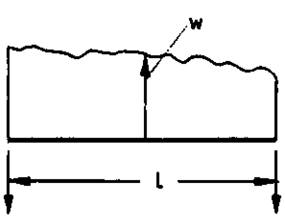
B 4.2.4 <u>The Three-Moment Equation (Cont'd)</u>

Table B 4.2.4-1 K_1 and K_2 Values for Various Load Conditions

Type of Loading	Left Bay $-K_1$	Right Bay $- K_2$
1. 	$\dfrac{+w_1L_1^3}{4I_1}$	$\dfrac{+w_2L_2^3}{4I_2}$
2. 	$\dfrac{+w_1c_1\ (3L_1^2-C_1^2)}{8I_1}$	$\dfrac{+w_2c_2\ (3L_2^2-c_2^2)}{8I_2}$
3. 	$K_1 = + \dfrac{w_1\left[b_1^2(2L_1^2-b_1^2)-a_1^2(2L_1^2-a_1^2)\right]}{4I_1L_1}$ $K_2 = + \dfrac{w_2\left[b_2^2(2L_2^2-b_2^2)-a_2^2(2L_2^2-a_2^2)\right]}{4I_2L_2}$	
4. 	$\dfrac{+2w_1L_1^3}{15I_1}$	$\dfrac{+7w_2L_2^3}{60\ I_2}$
5. 	$\dfrac{+7w_1L_1^3}{60I_1}$	$\dfrac{+2w_2L_2^3}{15I_2}$
6. 	$\dfrac{+3L_1^2w_1}{8I_1}$	$\dfrac{+3L_2^2w_2}{8I_2}$

B 4.2.4 The Three-Moment Equation (Cont'd)

Table 4.2.4-1 K_1 and K_2 Values for Various Load Conditions

Type of Loading	Left Bay - K_1	Right Bay - K_2
7. 	$+\frac{5w_1L_1^3}{32I_1}$	$+\frac{5w_2L_2^2}{32I_2}$
8. 	$+\frac{w_1a_1\left(L_1^2-a_1^2\right)}{I_1L_1}$	$+\frac{w_2b_2\left(L_2^2-b_2^2\right)}{I_2L_2}$
9. 	$+\frac{M_1}{I_1}\left(L_1-\frac{3a_1^2}{L_1}\right)$	$+\frac{M_2}{I_2}\left(\frac{3b_2^2}{L_2}-L_2\right)$
10. 	$K_1 = \frac{6}{I_2L_2}\int_o^L M_1\, x_1\, dx_1$ $K_2 = \frac{6}{I_2L_2}\int_o^L M_2\, x_2\, dx_2$ Where M is the bending moment.	

B 4.2.5 <u>Moment Distribution Method</u>

The moment distribution method is suggested for the solution of problems involving continuous beams with many redundant supports. The discussion and application of the method is given in section B 5.0.0 (Frames).

B 4.3.0 Curved Beams

B 4.3.1 Correction Factors for Use in Straight-Beam Formula

When a curved beam is bent in the plane of initial curvature, plane sections remain plane, but the strains of the fibers are not proportional to the distance from the neutral axis because the fibers are not at equal length. If (K) denotes a correction factor, the stress at the extreme fiber of a curved beam is given by

$$f = K \frac{Mc}{I}$$

in which

$$K = \frac{\frac{M}{AR}\left[1 + \frac{c}{Z(R + c)}\right]}{\frac{Mc}{I}}$$

where

M is the bending moment
A is the cross sectional area
R is the radius of curvature to the centroidal axis
c is the distance from the centroidal axis to the extreme outer fiber
I is the moment of inertia

$$Z = -\frac{1}{A}\int \frac{y}{R+y}\, dA$$

Values of K for different sections are given in Table B 4.3.1.1.

Table B 4.3.1-1

Values of K for Different Sections and Different Radii of Curvature.

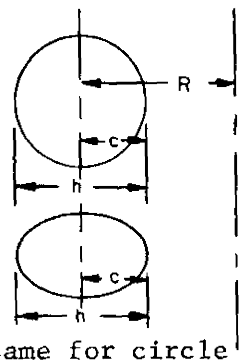
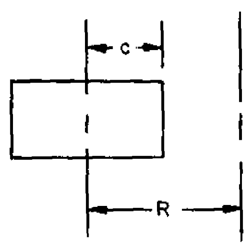
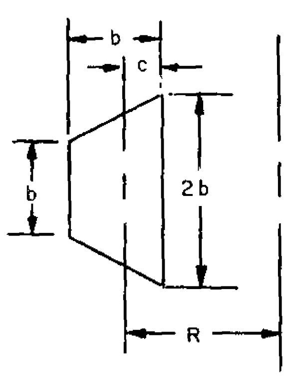
Section	$\frac{R}{c}$	Factor K	
		Inside Fiber	Outside Fiber
1.  K the same for circle and ellipse and independent of dimensions.	1.2	3.41	0.54
	1.4	2.40	0.60
	1.6	1.96	0.65
	1.8	1.75	0.68
	2.0	1.62	0.71
	3.0	1.33	0.79
	4.0	1.23	0.84
	6.0	1.14	0.89
	8.0	1.10	0.91
	10.0	1.08	0.93
2.  K independent of section dimensions	1.2	2.89	0.57
	1.4	2.13	0.63
	1.6	1.79	0.67
	1.8	1.63	0.70
	2.0	1.52	0.73
	3.0	1.30	0.81
	4.0	1.20	0.85
	6.0	1.12	0.90
	8.0	1.09	0.92
	10.0	1.07	0.94
3. 	1.2	3.01	0.54
	1.4	2.18	0.60
	1.6	1.87	0.65
	1.8	1.69	0.68
	2.0	1.58	0.71
	3.0	1.33	0.80
	4.0	1.23	0.84
	6.0	1.13	0.88
	8.0	1.10	0.91
	10.0	1.08	0.93

Table B 4.3.1-1 (Cont'd)

Values of K for Different Sections and Different Radii of Curvature

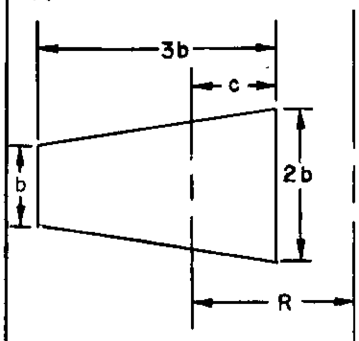
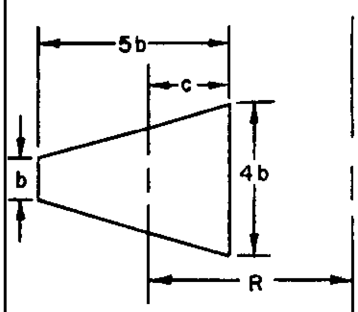
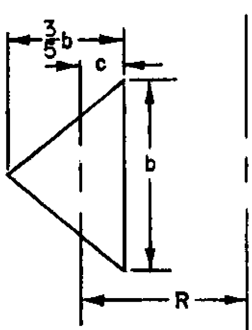
Section	$\frac{R}{c}$	Factor K Inside Fiber	Factor K Outside Fiber
4. 	1.2	3.09	0.56
	1.4	2.25	0.62
	1.6	1.91	0.66
	1.8	1.73	0.70
	2.0	1.61	0.73
	3.0	1.37	0.81
	4.0	1.26	0.86
	6.0	1.17	0.91
	8.0	1.13	0.94
	10.0	1.11	0.95
5. 	1.2	3.14	0.52
	1.4	2.29	0.54
	1.6	1.93	0.62
	1.8	1.74	0.65
	2.0	1.61	0.68
	3.0	1.34	0.76
	4.0	1.24	0.82
	6.0	1.15	0.87
	8.0	1.12	0.91
	10.0	1.10	0.93
6. 	1.2	3.26	0.44
	1.4	2.39	0.50
	1.6	1.99	0.54
	1.8	1.78	0.57
	2.0	1.66	0.60
	3.0	1.37	0.70
	4.0	1.27	0.75
	6.0	1.16	0.82
	8.0	1.12	0.86
	10.0	1.09	0.88

Table B 4.3.1-1 (Cont'd)

Values of K for Different Sections and Different Radii of Curvature

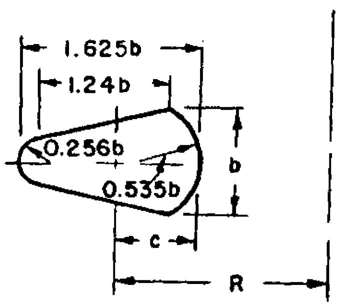
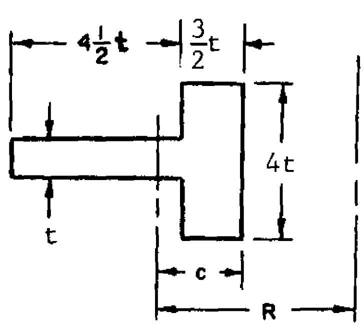
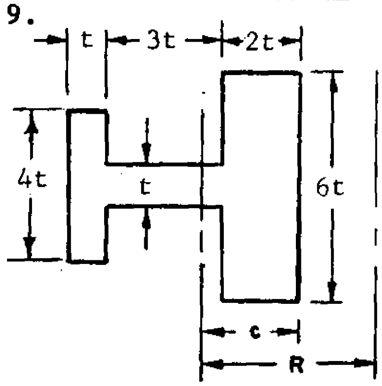
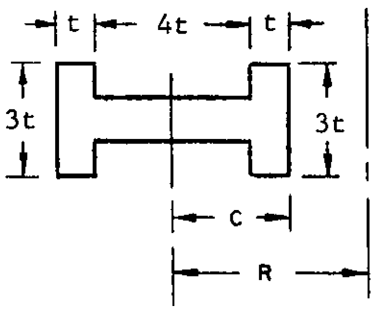
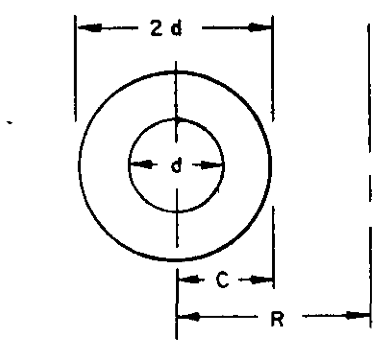
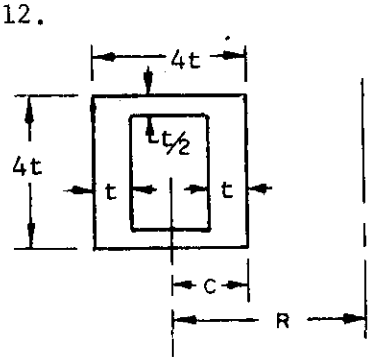
Section	$\frac{R}{c}$	Factor K	
		Inside Fiber	Outside Fiber
7. 	1.2	3.65	0.53
	1.4	2.50	0.59
	1.6	2.08	0.63
	1.8	1.85	0.66
	2.0	1.69	0.69
	2.5	1.49	0.74
	3.0	1.38	0.78
	4.0	1.27	0.83
	6.0	1.19	0.90
	8.0	1.14	0.93
$A = 1.05b^2$ $I = 0.18b^4$ $C = 0.70b$	10.0	1.12	0.96
8. 	1.2	3.63	0.58
	1.4	2.54	0.63
	1.6	2.14	0.67
	1.8	1.89	0.70
	2.0	1.73	0.72
	3.0	1.41	0.79
	4.0	1.29	0.83
	6.0	1.18	0.88
	8.0	1.13	0.91
	10.0	1.10	0.92
9. 	1.2	3.55	0.67
	1.4	2.48	0.72
	1.6	2.07	0.76
	1.8	1.83	0.78
	2.0	1.69	0.80
	3.0	1.38	0.86
	4.0	1.26	0.89
	6.0	1.15	0.92
	8.0	1.10	0.94
	10.0	1.08	0.95

Table B 4.3.1-1 (Cont'd)

Values of K for Different Sections and Different Radii of Curvature

Section	R/c	Factor K	
		Inside Fiber	Outside Fiber
10. 	1.2	2.52	0.67
	1.4	1.90	0.71
	1.6	1.63	0.75
	1.8	1.50	0.77
	2.0	1.41	0.79
	3.0	1.23	0.86
	4.0	1.16	0.89
	6.0	1.10	0.92
	8.0	1.07	0.94
	10.0	1.05	0.95
11. 	1.2	3.28	0.58
	1.4	2.31	0.64
	1.6	1.89	0.68
	1.8	1.70	0.71
	2.0	1.57	0.73
	3.0	1.31	0.81
	4.0	1.21	0.85
	6.0	1.13	0.90
	8.0	1.10	0.92
	10.0	1.07	0.93
12. 	1.2	2.63	0.68
	1.4	1.97	0.73
	1.6	1.66	0.76
	1.8	1.51	0.78
	2.0	1.43	0.80
	3.0	1.23	0.86
	4.0	1.15	0.89
	6.0	1.09	0.92
	8.0	1.07	0.94
	10.0	1.06	0.95

B 4.4.0 Bending-Crippling Failure of Formed Beams

This section contains methods of analysis applicable to formed or built-up sections which are critical in the bending-crippling mode of failure. This method is to be used when plastic bending curves are not available, otherwise use Section B4.5.

It is noted that a positive margin of safety derived from this analysis does not preclude failure in another mode.

The analysis procedure is divided into two sections according to the type of applied loading as follows:

Section

B 4.4.1 Bending moment only.
B 4.4.2 Combined bending moment and axial load.

Examples are given to illustrate the procedure for each type of loading.

B 4.4.1 Analysis Procedure for Bending Moment Only

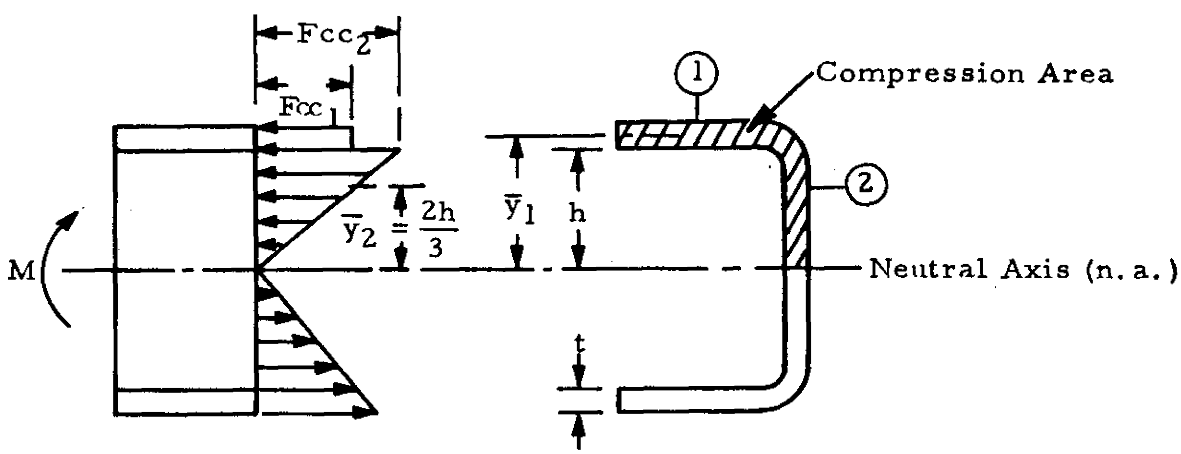


Figure 1. Bending-Crippling of Formed Shapes

B4.4.1 Analysis Procedure for Bending Moment Only (Cont'd)

(a) Locate the neutral axis (line of zero fiber stress) of the cross-section assuming a linear stress distribution.
(b) Divide the compression area into elements according to Section C 1, pages 11-16.
(c) Calculate $Fcc_{n,m}$ according to Figure C 1.3.1-13 for each element.
(d) Calculate the allowable bending-crippling moment by summing moments about the neutral axis for the compression area and doubling the result.

$$\overline{M} = 2 \left[\underbrace{\Sigma\, Fcc_n\, b_n\, t_n\, \overline{y}_n}_{\text{For flange members}} + \underbrace{\Sigma\, Fcc_m\, b_m\, t_m\, \overline{y}_m/2}_{\text{For web members}} \right] \quad \ldots\ldots\ldots\ldots (1)$$

where:

$\overline{y}_{n,m}$ = distance from neutral axis to the resultant force of each element.

This equation is applicable for all shapes although only a formed channel is shown in Figure 1.

(e) The margin of safety is given by:

$$\text{(ult) M.S.} = \frac{\overline{M}}{(FS)_{ult}\, M} - 1 \quad \ldots\ldots\ldots\ldots (2)$$

where:

M = applied bending moment at the cross-section in question.

$\overline{M}$ = allowable bending-crippling moment from Eq. (1).

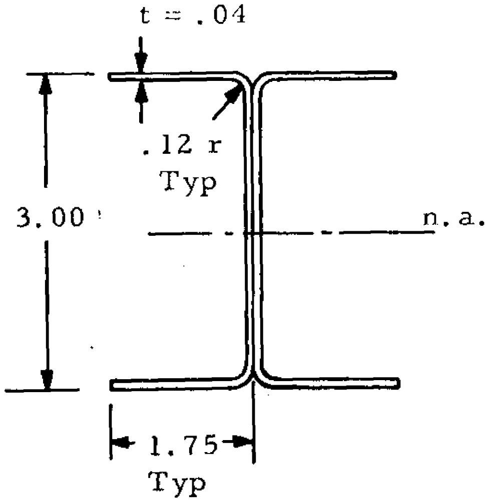
$(FS)_{ult}$ = ultimate factor of safety.

Note: if the section is unsymmetrical, the tension flange should be analyzed in the conventional manner.

B4.4.1 Analysis Procedure for Bending Moment Only (Cont'd)

Example 1

Determine the margin of safety in bending-crippling for the cross-section shown below if the bending moment is 4000 in.-lb and a factor of safety of 1.4 is desired.



Given:

Mat'l = 6061-T6 Bare Sht.

Mech. Prop.

F_{tu} = 42 ksi

F_{cy} = 35 ksi

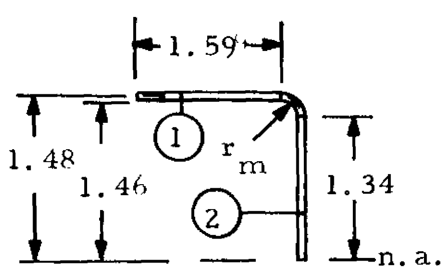
$E = 9.9 \times 10^3$ ksi

Channels are intermittently riveted together.

Figure 2. Back-to-Back Formed Channels

Analysis

(a) The neutral (centroidal) axis for this case is determined by inspection.



$r_m = .12 + .02 = .14$ in.

$b_1 = 1.59 + .535(.14) = 1.665$ in.

$b_2 = 1.34 + .535(.14) = 1.415$ in.

B4.4.1 Analysis Procedure for Bending Moment Only (Cont'd)

Example 1 (Cont'd)

(c) $$\sqrt{\frac{F_{cy}}{E}}\ \frac{b_1}{t} = \sqrt{\frac{35}{9.9 \times 10^3}}\ \frac{1.665}{.040} = 2.475, \quad Fcc_1 = .275(35) = 9.62 \text{ ksi}$$

$$\sqrt{\frac{F_{cy}}{E}}\ \frac{b_2}{t} = \sqrt{\frac{35}{9.9 \times 10^3}}\ \frac{1.415}{.040} = 2.103, \quad Fcc_2 = .76(35) = 26.6 \text{ ksi}$$

(d) $\overline{M} = 2[M_{flange} + M_{web}]$ Ref. Eq. (1)

$$= 2\,[(9.62)(1.665)(.04)(1.48) + (26.6)(1.415)(.04)(1.46)/3]$$

$= 3.36$ in-kips (for each channel) See page 278 for Fcc_n, b_n, t_n, and $\overline{y}_n$

(e) Margin of safety

$$\text{(ult) M.S.} = \frac{\overline{M}}{(FS)_{ult}\ M/2} - 1 = \frac{3360}{1.4(2000)} - 1 = \underline{\underline{+\,0.20}}$$

where:

$M/2$ = 2000 in-lb (per channel)

$(FS)_{ult} = 1.4$

Ref. page 278

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load

1. Calculate steps (a) through (d) according to the procedure of Section B 4.6.1. If the neutral axis falls outside the cross-section, consider the section to be stressed as a column and compare with the maximum applied fiber stress.

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

2. Calculate the section modulus of the compression area about the neutral axis assuming a mirror image.

$$Z_{c_{n.a.}} = \frac{I_{n.a.}}{C_c} \text{ (see Example 2)} \quad \ldots\ldots \quad (3)$$

3. Compute the equivalent allowable stress

$$\overline{F} = \frac{\overline{M}}{Z_{c_{n.a.}}} \quad \ldots\ldots \quad (4)$$

4. The margin of safety is given by

$$\text{(ult) MS} = \frac{\overline{F}}{(FS)_{ult}\, f_c} - 1 \quad \ldots\ldots \quad (5)$$

where:

$$f_c = \frac{P}{A} \pm \frac{Mc}{I_{c.g.}} \text{ (maximum applied compressive stress)}$$

Note: If the normal load (P) is tensile, the tension flange should be analyzed in the conventional manner.

Example 2

Determine the margin of safety for bending-crippling failure for the beam column shown in Figure 3.

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

Example 2 (Cont'd)

Given:

Mat'l = 6061-T6 Bare Sht   P = 5000 lb

Mech. Prop.   M = 6400 in-lb

F_{tu} = 42 ksi $(FS)_{ult}$ = 1.4

F_{cy} = 35 ksi

E = 9.9 x 10^3 ksi

Angles are intermittently riveted together.

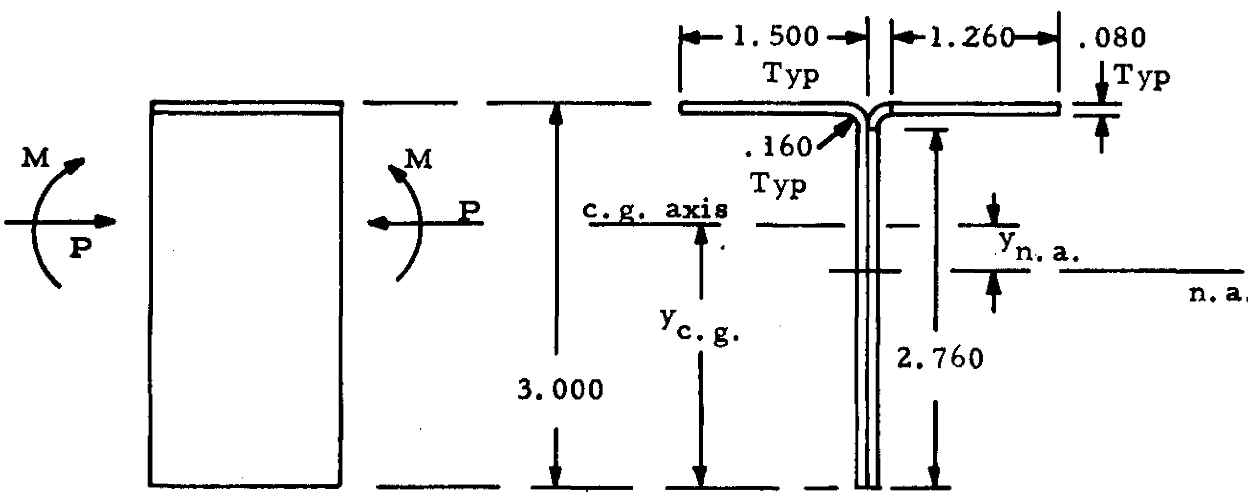
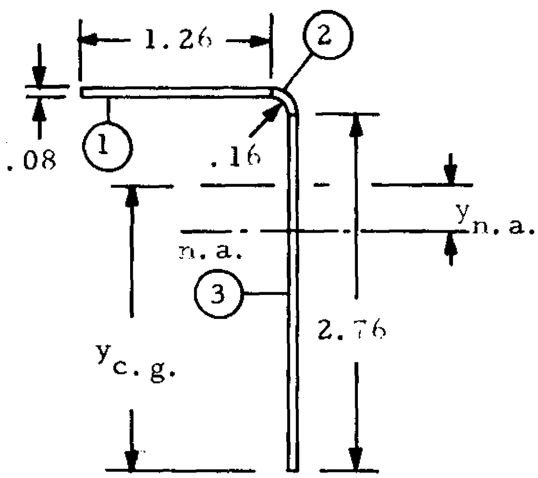


Figure 3. Back-to-Back Formed Angle

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

Example 2 (Cont'd)

1.



ELE	A	y	Ay	Ay^2	I_o
①	.1008	2.96	.298	.883	-
②	.0250	2.89	.072	.209	-
③	.2208	1.38	.305	.420	.140
TOT.	.3466		.675	1.512	.140

$$y_{c.g.} = \frac{\Sigma Ay}{\Sigma A} = \frac{.675}{.3466} = 1.947 \text{ in.}$$

$$I_{c.g.} = \Sigma Ay^2 + \Sigma I_o - y_{c.g.}^2 \Sigma A = 1.512 + .140 - (1.947)^2 (.3466)$$

$$= .338 \text{ (for each angle)}$$

$$0 = \frac{P}{A} \pm \frac{My_{n.a.}}{I_{c.g.}}$$

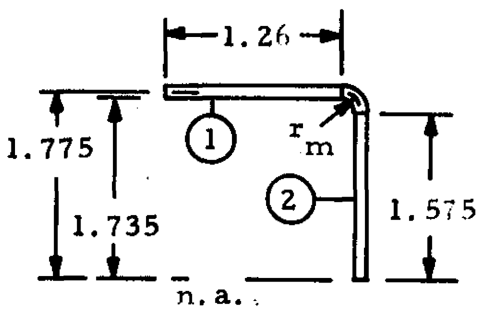
(a)

$$y_{n.a.} = -\frac{P}{A}\frac{I_{c.g.}}{M} = -\frac{5000\,(2)(.338)}{2(.3466)\,6400} = -.762 \text{ in.}$$

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

Example 2 (Cont'd)

(b)



$r_m = .16 + .04 = .20$ in.

$b_1 = 1.26 + .535(.20) = 1.367$ in.

$b_2 = 1.575 + .535(.20) = 1.682$ in.

(c)

$$\sqrt{\frac{F_{cy}}{E}}\ \frac{b_1}{t} = \sqrt{\frac{35}{9.9 \times 10^3}}\ \frac{1.367}{.08} = 1.016,\quad Fcc_1 = .56(35) = 19.60 \text{ ksi}$$

$$\sqrt{\frac{F_{cy}}{E}}\ \frac{b_2}{t} = \sqrt{\frac{35}{9.9 \times 10^3}}\ \frac{1.682}{.08} = 1.250,\quad Fcc_2 = 1.1(35) = 38.5 \text{ ksi}$$

(d)

$$\overline{M} = 2[M_{flange} + M_{web}] \quad \text{Ref. Eq. (1)}$$

$$= 2[19.60(1.367)(.08)(1.775) + 38.5(1.682)(.08)(1.735)/3]$$

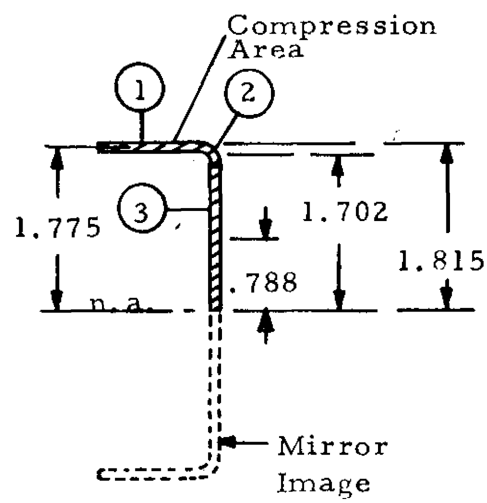
$$= 13.60 \text{ in-kip} \quad \text{(each angle)}$$

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

Example 2 (Cont'd)

2. Section modulus about n.a.

ELE	A	y	Ay	Ay^2	I_o
①	.1008	1.775	.179	.318	-
②	.0250	1.702	.042	.072	-
③	.1260	.788	.099	.078	.026
TOT.				.468	.026



$$I_{n.a.} = 2[\Sigma Ay^2 + \Sigma I_o]$$

$$= 2[.468 + .026] = .988 \text{ in}^4 \text{ (for each angle)}$$

$$Z_{c_{n.a.}} = \frac{2(.988)}{1.815} = 1.089 \text{ in}^3$$

3. Equivalent allowable stress

$$\overline{F} = \frac{2\ (\overline{M})}{Z_{c_{n.a.}}} = \frac{2(13.60)}{1.089} = 25.0 \text{ ksi}$$

B4.4.2 Analysis Procedure for Combined Bending Moment and Axial Load (Cont'd)

Example 2 (Cont'd)

Applied compressive stress

$$f_c = \frac{P}{A} + \frac{Mc}{I_{c.g.}}$$

$$= \frac{5}{2(.3466)} + \frac{6.4(1.053)}{2(.338)} = 17.2 \text{ ksi}$$

4. The margin of safety is

$$(ult)\ MS = \frac{\overline{F}}{(FS)_{ult}\ f_c} - 1 = \frac{25.0}{1.4(17.2)} - 1 = +\underline{\underline{0.04}}$$

where:

$(FS)_{ult} = 1.4$ Ref. page 281

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