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VOLUME I

Structures and Propulsion Laboratory

August 1975

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16. ABSTRACT

This document (Volumes I, II, and III) presents a compilation of industry-wide methods in aerospace strength analysis that can be carried out by hand, that are general enough in scope to cover most structures encountered, and that are sophisticated enough to give accurate estimates of the actual strength expected. It provides analysis techniques for the elastic and inelastic stress ranges. It serves not only as a catalog of methods not usually available, but also as a reference source for the background of the methods themselves.

An overview of the manual is as follows: Section A is a general introduction of methods used and includes sections on loads, combined stresses, and interaction curves; Section B is devoted to methods of strength analysis; Section C is devoted to the topic of structural stability; Section D is on thermal stresses; Section E is on fatigue and fracture mechanics; Section F is on composites; Section G is on rotating machinery; and Section H is on statistics.

These three volumes supersede NASA TM X-60041 and NASA TM X-60042.

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STRUCTURES MANUAL

FOREWORD

This manual is issued to the personnel of the Strength Analysis Branch to provide uniform methods of structural analysis and to provide a ready reference for data. Generally, the information contained in this manual is a condensation of material published by universities, scientific journals, missile and aircraft industries, text book publishers, and government agencies.

Illustrative problems to clarify either the method of analysis or the use of the curves and tables are included wherever they are considered necessary. Limitations of the procedures and the range of applicability of the data are indicated wherever possible.

It is recognized that all subjects in the Table of Contents are not present in the body of the manual; some sections remain to be developed in the future. However, an alphabetical index of content material is provided and is updated as new material is added. New topics not listed in the Table of Contents will be treated as the demand arises. This arrangement has been utilized to make a completed section available as soon as possible. In addition, revisions and supplements are to be incorporated as they become necessary.

Many of the methods included have been adapted for computerized utilization. These programs are written in Fortran Language for utilization on the MSFC Executive VIII, Univac 1108, or IBM 7094 and are cataloged with example problems in the Structural Analysis Computer Utilization Manual.

It is requested that any comments concerning this manual be directed to:

Chief, Structural Requirements Section
Strength Analysis Branch
Analytical Mechanics Division
Astronautics Laboratory
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama 35812

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ASTRONAUTICS STRUCTURES MANUAL

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SECTION A1

STRESS AND STRAIN

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A1.0.0 Stress and Strain

The relationship between stress and strain and other material properties, which are used throughout this manual, are presented in this section. A brief introduction to the theory of elasticity for elementary applications is also presented in this section.

A1.1.0 Mechanical Properties of Materials

A brief account of the important mechanical properties of materials is given in this subsection; a more detailed discussion may be found in any one of a number of well known texts on the subject. The numerical values of the various mechanical properties of most aerospace materials are given in MIL-HDBK-5 (reference 1). Many of these values are obtained from a plotted set of test results of one type or another. One of the most common sets of these plotted sets is the stress-strain diagram. A typical stress-strain diagram is discussed in the next subsection.

A1.1.1 Stress-Strain Diagram

Some of the more useful properties of materials are obtained from a stress-strain diagram. A typical stress-strain curve for aerospace metals is shown in Figure A1.1.1-1.

The curve in Figure A1.1.1-1 is composed of two regions; the straight line portion up to the proportional limit where the stress varies linearly with strain, and the remaining part where the stress is not proportional to strain. In this manual, stresses below the ultimate tensile stress (F_{tu}) are considered to be elastic. However, a correction (or plasticity reduction) factor is sometimes employed in certain types of analysis for stresses above the proportional limit stress.

Commonly used properties shown on a stress-strain curve are described briefly in the following paragraphs:

E — Modulus of elasticity; average ratio of stress to strain for stresses below the proportional limit. In Figure A1.1.1-1 $E = \tan \theta$

A1.1.1 Stress-Strain Diagram (Cont'd)

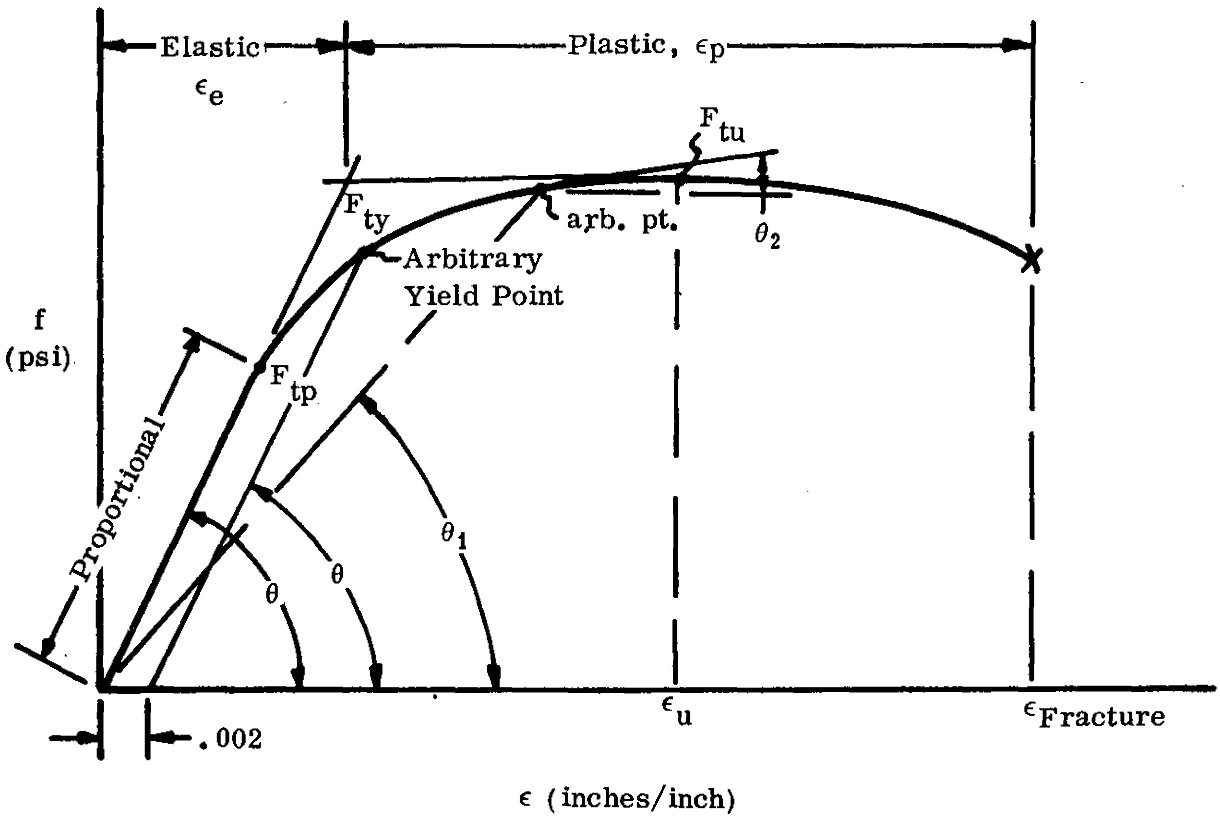

Figure A1.1.1-1 A Typical Stress-Strain Diagram

E_s — Secant modulus; ratio of stress to strain above the proportional limit; reduces to E in the proportional range. In Figure A1.1.1-1 $E_s = \tan \theta_1$

E_t — Tangent modulus; slope of the stress-strain curve at any point; reduces to E in the proportional range. In Figure A1.1.1-1 $E_t = \frac{df}{d\epsilon} = \tan \theta_2$.

A1.1.1 Stress-Strain Diagram (Cont'd)

F_{ty} or F_{cy}	Tensile or compressive yield stress; since many materials do not exhibit a definite yield point, the yield stress is determined by the .2% offset method. This entails the construction of a straight line with a slope E passing through a point of zero stress and a strain of .002 in./in. The intersection of the stress-strain curve and the constructed straight line defines the magnitude of the yield stress.
F_{tp} or F_{cp}	Proportional limit stress in tension or compression; the stress at which the stress ceases to vary linearly with strain.
F_{tu}	Ultimate tensile stress; the maximum stress reached in tensile tests of standard specimens.
F_{cu}	Ultimate compressive stress; taken as F_{tu} unless governed by instability.
ϵ_u	The strain corresponding to F_{tu}.
ϵ_e	Elastic strain; see Figure A1.1.1-1.
ϵ_p	Plastic strain; see Figure A1.1.1-1.
$\epsilon_{fracture}$ (% elongation)	Fracture strain; percent elongation in a predetermined gage length associated with tensile failures, and is a relative indication of ductility of the material.

A1.1.2 Other Material Properties

The definition of various other material properties and terminology used in stress analysis work is given in this subsection.

A1.1.2 Other Material Properties (Cont'd)

F_{bry}, F_{bru}	Yield and ultimate bearing stress; determined in a manner similar to those for tension and compression. A load-deformation curve is plotted where the deformation is the change in the hole diameter. Bearing yield (F_{bry}) is defined by an offset of 2% of the hole diameter; bearing ultimate (F_{bru}) is the actual failing stress divided by 1.15.
F_{su}	Ultimate shear stress.
F_{sp}	Proportional limit in shear; usually taken equal to 0.577 times the proportional limit in tension for ductile materials.
ν	Poisson's ratio; the ratio of transverse strain to axial strain in a tension or compression test. For materials stressed in the elastic range, ν may be taken as a constant but for inelastic strains ν becomes a function of axial strain.
ν_p	Plastic Poisson's ratio; unless otherwise stated, ν_p may be taken as 0.5.
$G = \frac{E}{2(1+\nu)}$	Modulus of rigidity or shearing modulus of elasticity for pure shear in isotropic materials.
Isotropic	Elastic properties are the same in all directions.
Anisotropic	Elastic properties differ in different directions.
Orthotropic	Distinct material properties in mutually perpendicular planes.

A1.1.3 Strain-Time Diagram

The behavior of a structural material is dependent on the duration of loading. This behavior is exhibited with the aid of a strain-time diagram such as that shown in Figure A1.1.3-1. This diagram consists of regions that are dependent

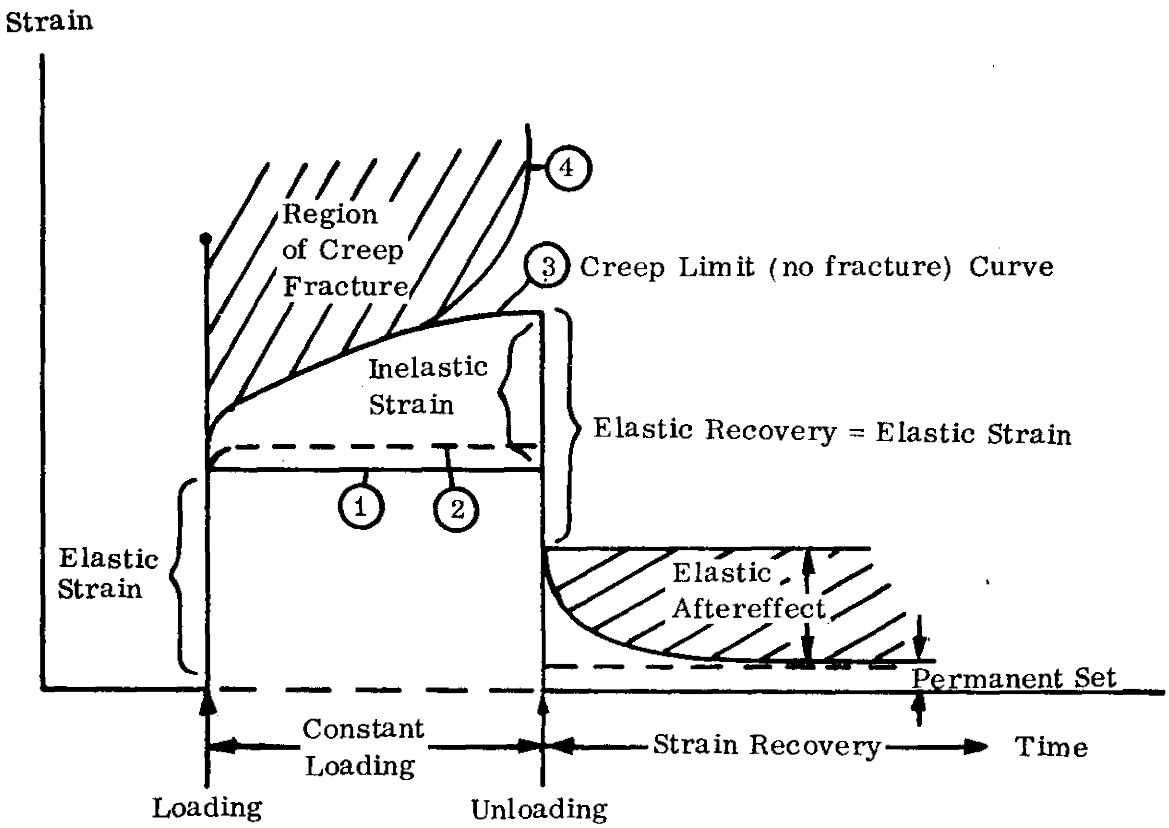

Figure A1.1.3-1 Strain-Time Diagram

upon the four loading conditions as indicated on the time coordinate. These loading conditions are as follows:

1. Loading

2. Constant loading

A1.1.3 Strain-Time Diagram (Cont'd)

3. Unloading

4. Recovery (no load)

The interval of time when the load is held constant is usually measured in weeks or months. Whereas the time involved in loading and unloading is relatively short (usually seconds or minutes) such that the corresponding strain-time curve can be represented by a straight vertical line.

The following discussion of the diagram will be confined to generalities due to the complexity of the phenomena of creep and fracture. A more detailed discussion on this subject is presented in reference 5.

The condition referred to as "loading" represents the strain due to a load which is applied over a short interval of time. This strain may vary from zero to the strain at fracture ($\epsilon_{fracture}$ - See Figure A1.1.1-1) depending upon the material and loading.

During the second loading condition, where the load is held constant, the strain-time curve depends on the initial strain for a particular material. The possible strain-time curves (Figure A1.1.3-1) that could result are discussed below.

a. In curve 1, the initial strain is elastic and no additional strain is experienced for the entire time interval. This curve typifies elastic action.

b. In curve 2, the initial strain increases for a short period after the load becomes constant and then remains constant for the remainder of the period. This action is indicative of slip which is characterized by a permanent set resulting from the shifting (slip) of adjacent crystalline structures along planes most favorably oriented with respect to the direction of the principal shearing stress.

c. In curve 3, there is a continuous increase in strain after the initial slip until a steady state condition is attained. This curve is indicative of creep which is generally the result of a combined effect of the predominantly viscous inelastic deformation within the unordered intercrystalline boundaries and the complex deformations by slip and fragmentation of the ordered crystalline domains.

A1.1.3 Strain-Time Diagram (Cont'd)

d. Curve 4 is also a combination of slip and creep. The only difference from curve 3 is that the creep action continues until the material fails in fracture. This fracture may take place at any time during the constant load period and is indicated by the upper shaded area in Figure A1.1.3-1.

During unloading, the reduction in strain of curves 1, 2 and 3 is equal to the elastic strain incurred during loading. This reduction is referred to as the "elastic recovery." It can be seen in Figure A1.1.3-1 that in the case of curve 1 the structural member will return to its initial configuration immediately after unloading. This is not the case for curves 2 and 3 as there will be some residual strain.

The last condition to be discussed on the strain-time diagram concerns the recovery period. In this period, some of the strain indicated as inelastic strain is recoverable. This is true particularly for many viscoelastic materials (such as flexible plastics) that do not show real creep, only delayed recoverable strains.

The height of the lower shaded area in Figure A1.1.3-1 is called the elastic after effect. The upper bound is the maximum possible permanent set and is indicated by the solid horizontal line. The lower bound could be any one of the family of possible strain-time curves confined within the lower shaded area. The limiting curve of the lower bound would approach the permanent set curve due to slip as indicated by the horizontal dashed line. If slip action is negligible, this limiting curve would be represented by a line that approaches zero asymptotically with increasing time.

A1.1.4 Temperature Effects

The mechanical properties of a material are usually affected by its temperature. This effect will be discussed in general terms in this section. For specific information, see the applicable chapter in reference 1.

In general, temperatures below room temperature increase the strength properties of metals. Ductility is usually decreased and the notch sensitivity of the metal may become of primary importance. The opposite is generally true for temperatures above room temperature.

A representative example for the effect of temperature on the mechanical properties of aluminum alloys is given in Figures A1.1.4-1 through 4. Most steels behave in a similar manner but generally are less sensitive to temperature magnitudes.

A1.1.4 Temperature Effects (Cont'd)

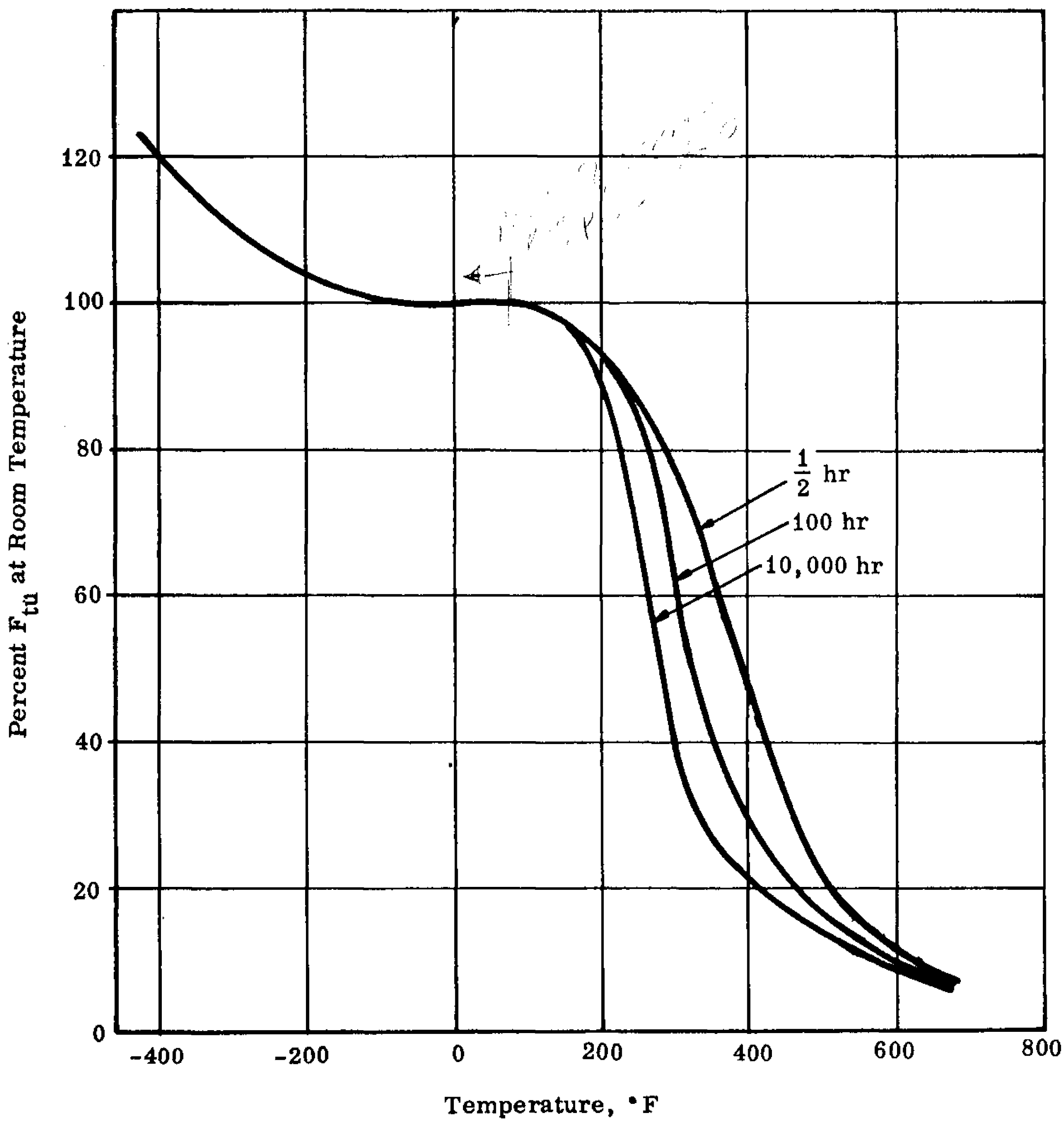

Figure A1.1.4-1 Effects of Temperature on the Ultimate Tensile Strength (F_{tu}) of 7079 Aluminum Alloy (from Ref. 1)

A1.1.4 Temperature Effects (Cont'd)

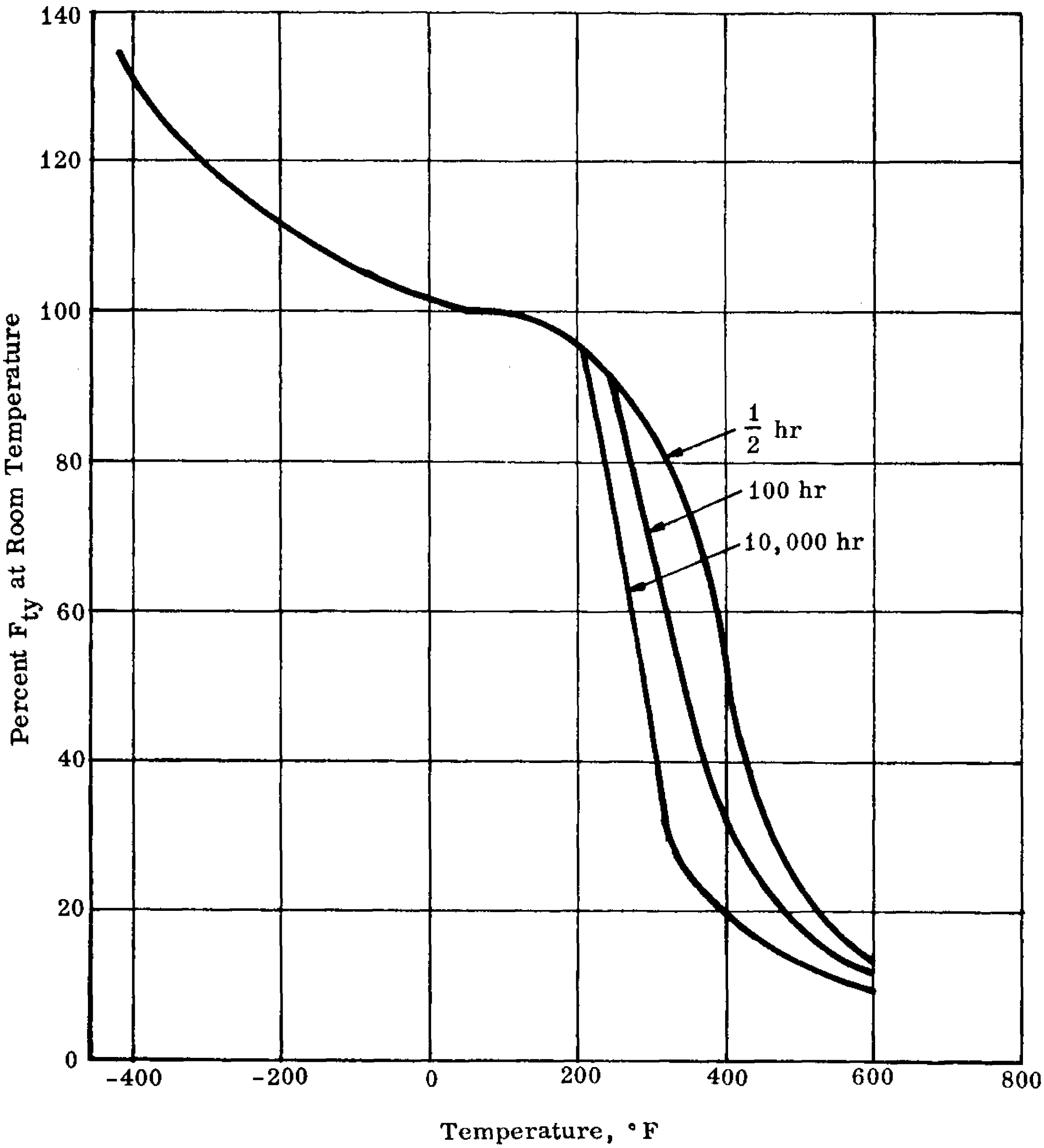

Figure A1.1.4-2 Effects of Temperature on the Tensile Yield Strength (F_{ty}) of 7079 Aluminum Alloy (from Ref. 1)

A1.1.4 Temperature Effects (Cont'd)

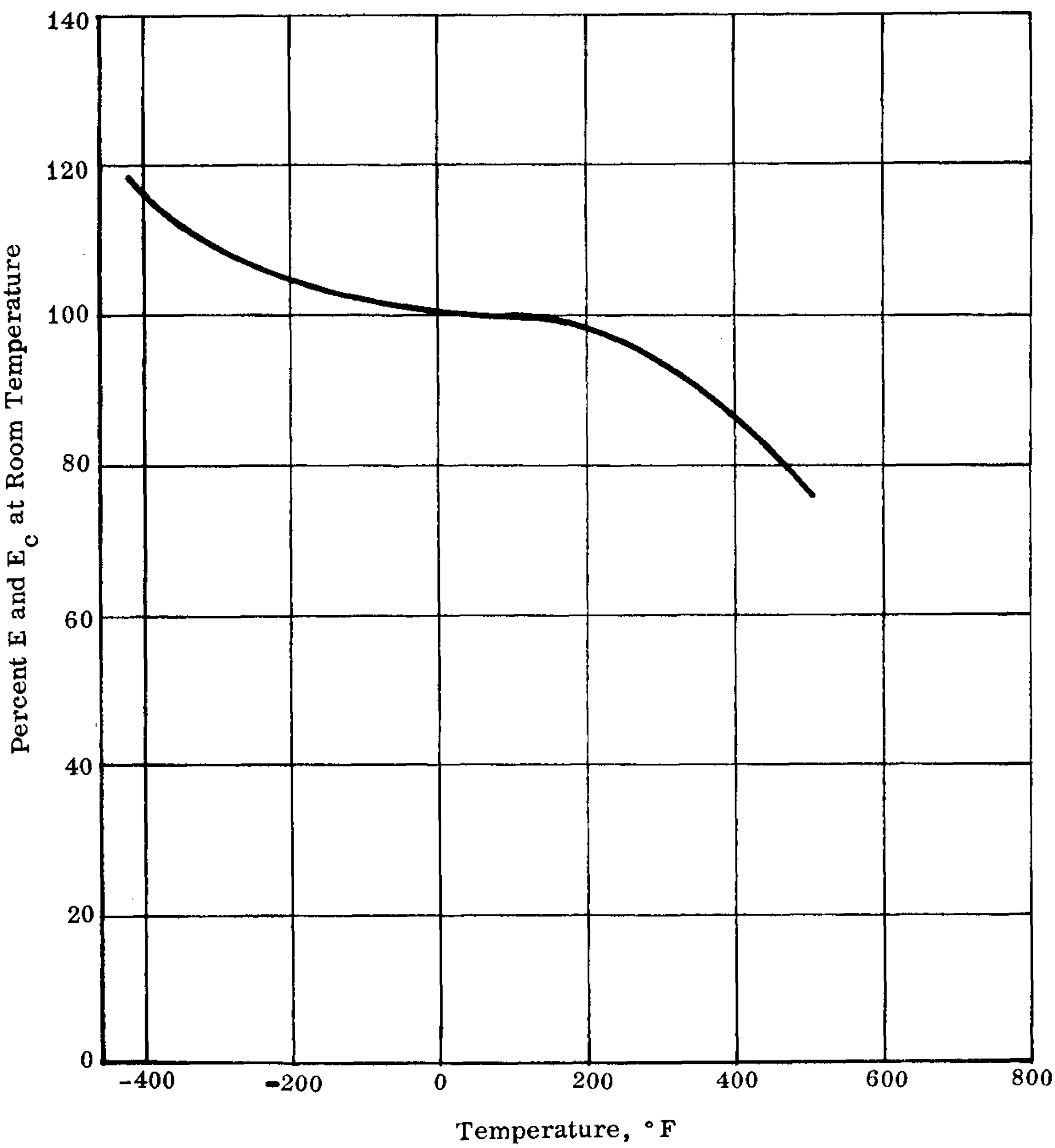

Figure A1.1.4-3 Effect of Temperature on the Tensile and Compressive Modulus (E and E_c) of 7079 Aluminum Alloy (from Ref. 1)

A1.1.4 Temperature Effects (Cont'd)

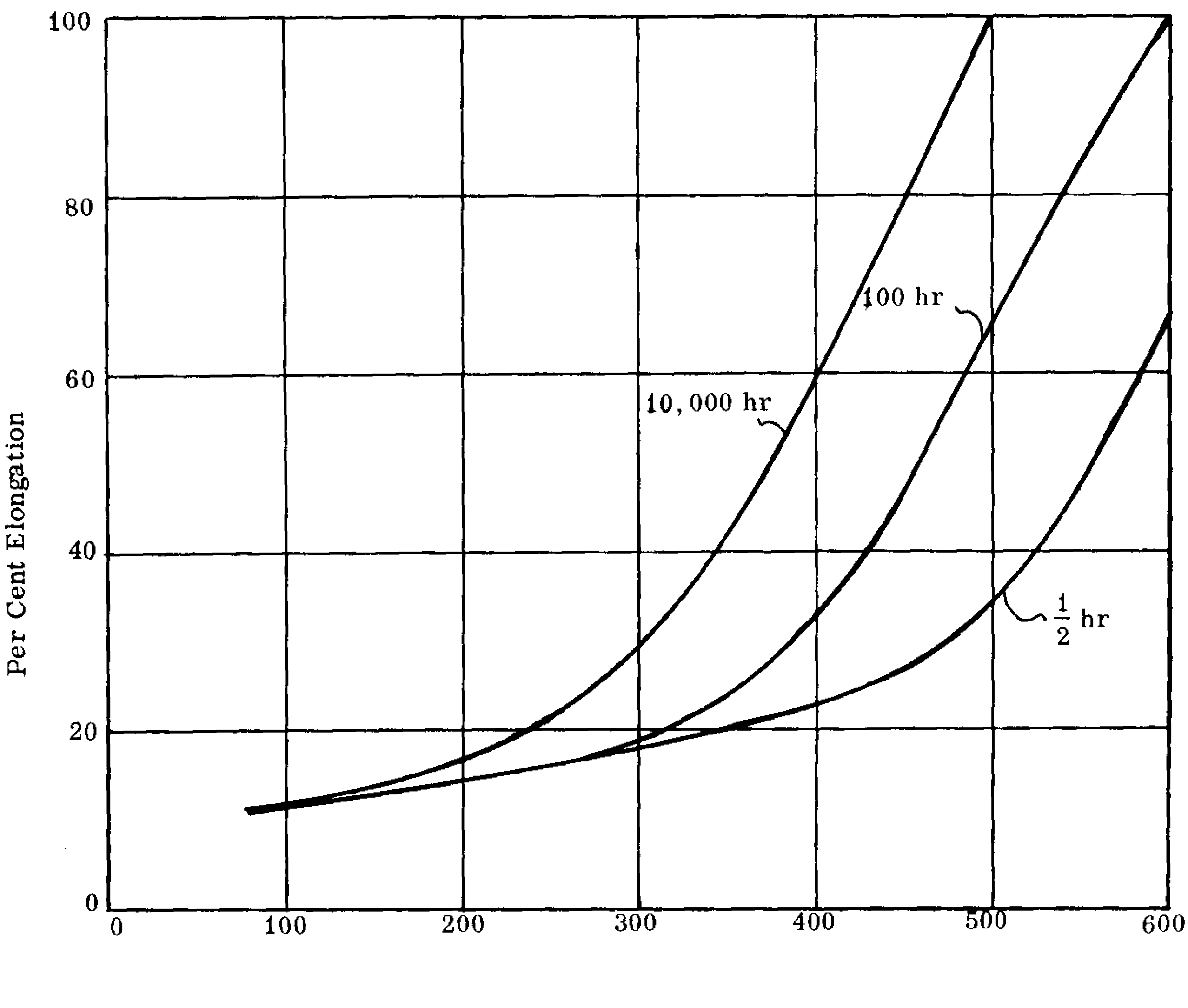

Figure A1.1.4-4 Effect of Temperature on the Elongation of 7079-T6 Aluminum Alloy (from Ref. 1)

A1.1.5 Hardness Conversion Table

A table for converting hardness numbers to ultimate tensile strength values is presented in this section. In this table, the ultimate strength values are in the range, 50 to 304 ksi. The corresponding hardness number is given for each of three hardness machines; namely, the Vickers, Brinell and the applicable scale(s) of the Rockwell machine.

This table is given in the remainder of this section. The appropriate materials-property handbook should be consulted for additional information whenever necessary.

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
50	104	92	--	58	--
52	108	96	--	61	--
54	112	100	--	64	--
56	116	104	--	66	--
58	120	108	--	68	--
60	125	113	--	70	--
62	129	117	--	72	--
64	135	122	--	74	--

Table A1.1.5-1 Hardness Conversion Table

A1.1.5 Hardness Conversion Table (Cont'd)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
66	139	127	--	76	--
68	143	131	--	77.5	--
70	149	136	--	79	--
72	153	140	--	80.5	--
74	157	145	--	82	--
76	162	150	--	83	--
78	167	154	51	84.5	--
80	171	158	52	85.5	--
82	177	162	53	87	--
83	179	165	53.5	87.5	--
85	186	171	54	89	--
87	189	174	55	90	--
89	196	180	56	91	--

Table A1.1.5-1 Hardness Conversion Table (Cont'd)

A1.1.5 Hardness Conversion Table (Cont'd)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
91	203	186	56.5	92.5	--
93	207	190	57	93.5	--
95	211	193	57	94	--
97	215	197	57.5	95	--
99	219	201	57.5	95.5	--
102	227	210	59	97	--
104	235	220	60	98	19
107	240	225	60.5	99	20
110	245	230	61	99.5	21
112	250	235	61.5	100	22
115	255	241	62	101	23
118	261	247	62.5	101.5	24
120	267	253	63	102	25

Table A1.1.5-1 Hardness Conversion Table (Cont'd)

A1.1.5 Hardness Conversion Table (Cont'd)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
123	274	259	63.5	103	26
126	281	265	64	--	27
129	288	272	64.5	--	28
132	296	279	65	--	29
136	304	286	65.5	--	30
139	312	294	66	--	31
142	321	301	66.5	--	32
147	330	309	67	--	33
150	339	318	67.5	--	34
155	348	327	68	--	35
160	357	337	68.5	--	36
165	367	347	69	--	37
170	376	357	69.5	--	38
176	386	367	70	--	39

Table A1.1.5-1 Hardness Conversion Table (Cont'd)

A1.1.5 Hardness Conversion Table (Cont'd)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
181	396	377	70.5	--	40
188	406	387	71	--	41
194	417	398	71.5	--	42
201	428	408	72	--	43
208	440	419	72.5	--	44
215	452	430	73	--	45
221	465	442	73.5	--	46
231	479	453	74	--	47
237	493	464	75	--	48
246	508	476	75.5	--	49
256	523	488	76	--	50
264	539	500	76.5	--	51
273	556	512	77	--	52
283	573	524	77.5	--	53

Table A1.1.5-1 Hardness Conversion Table (Cont'd)

A1.1.5 Hardness Conversion Table (Cont'd)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
294	592	536	78	--	54
304	611	548	78.5	--	55

Table A1.1.5-1 Hardness Conversion Table (Concluded)

A1.2.0 Elementary Theory of the Mechanics of Materials

In the elementary theory of mechanics of materials, a uni-axial state of strain is generally assumed. This state of strain is characterized by the simplified form of Hooke's law; namely $f = E\epsilon$, where ϵ is the unit strain in the direction of the unit stress f, and E is the Modulus of Elasticity. The strains in the perpendicular directions (Poisson's ratio effect) are neglected. This is generally justified in most elementary and practical applications considered in the theory of mechanics of materials. In these applications, the structural members are generally subjected to a uni-axial state of stress and/or the strains and displacements are of secondary importance. Also, in these applications, the magnitude of each of a set of bi-axial stresses (when this occurs) is generally independent of the Poisson's ratio effect.

Frequently in design, there are applications in which the magnitude of each of a set of bi-axial (or tri-axial) stresses are dependent upon the Poisson's ratio effect; and/or the magnitude of the strains and displacements are of primary importance. This type of application must be generally analyzed by the theory of elasticity. A brief account on the use of the theory of elasticity for elementary applications is given in the next subsection.

A1. 3. 0 Elementary Applications of the Theory of Elasticity

The difference between the method of ordinary mechanics and the theory of elasticity is that no simplifying assumption is made concerning the strains in the latter. Because of this, it becomes necessary to take into account the complete distribution of the strains in the body and to assume a more general statement of Hooke's law in expressing the relation between stresses and strains. It is noted that the stresses calculated by both methods are only approximate since the material in the physical body deviates from the ideal material assumed by both methods.

Some of the following subsections are written for a three dimensional stress field but are applicable to problems in two dimension simply by neglecting all terms containing the third dimension.

A1. 3. 1 Notation for Forces and Stresses

The stresses acting on the side of a cubic element can be described by six components of stress, namely the three normal stresses f_{11}, f_{22}, f_{33}, and the three shearing stresses $f_{12} = f_{21}$, $f_{13} = f_{31}$, $f_{23} = f_{32}$.

In Figure A1. 3. 1-1 shearing stresses are resolved into two components parallel to the coordinate axis. Two subscript numbers are used, the first indicating the direction normal to the plane under consideration and the second indicating the direction of the component of the stress. Normal stresses have like subscripts and positive directions are as shown in the figure. An analogous notation for the x-y coordinate system is:

$f_{11} = f_x$

$f_{22} = f_y$

$f_{12} = f_s$

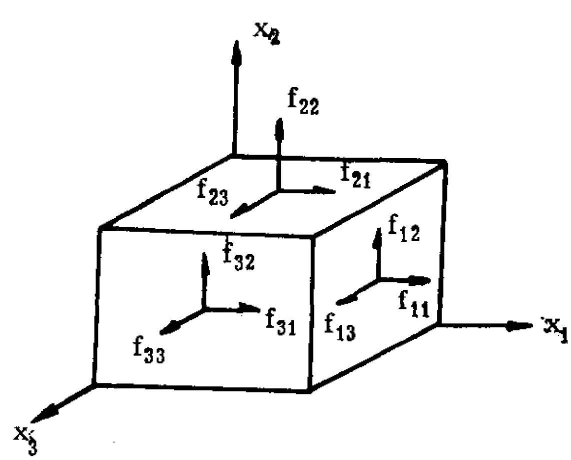

Figure A1. 3. 1-1 Representation of Stresses on an Element of a Body

A1.3.1 Notation for Forces and Stresses (Cont'd)

Surface forces

Forces distributed over the surface of the body, such as pressure of one body on another, or hydrostatic pressure, are called surface forces.

Body forces

Body forces are forces that are distributed over the volume of a body, such as gravitational forces, magnetic forces, or inertia forces in the case of a body in motion.

A1.3.2 Specification of Stress at a Point

If the components of stress in Figure A1.3.1-2 are known for any given point, the stress acting on any inclined plane through this point can be calculated from the equations of statics. Body forces, such as weight of the element, can generally be neglected since they are of higher order than surface forces.

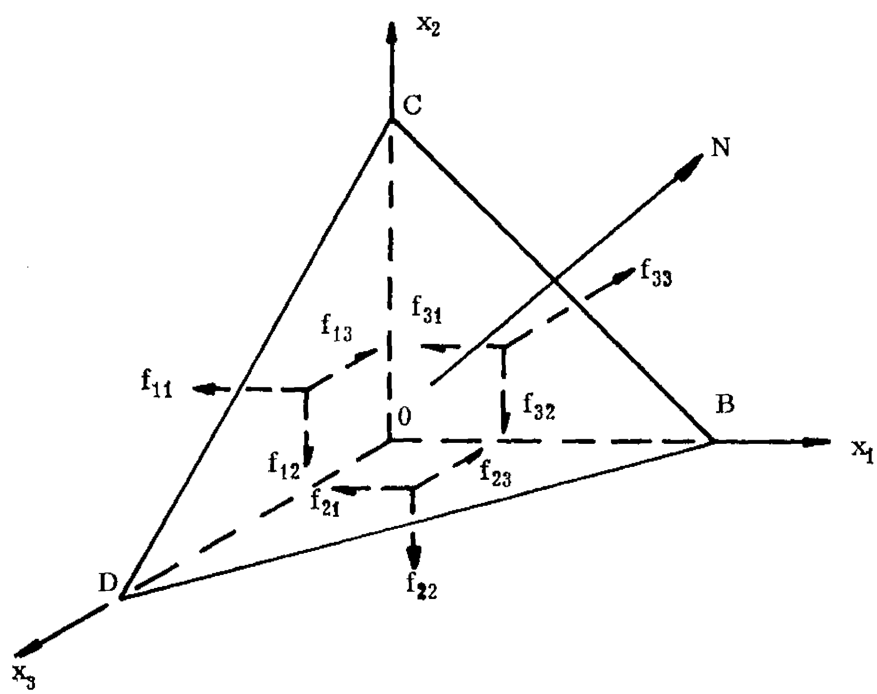

Figure A1.3.1-2 An Element Used in Specifying Stress at a Point

A1.3.2 Specification of Stress at a Point (Cont'd)

If A denotes the area of the inclined face BCD of the tetrahedron in Figure A1.3.1-2, then the areas of the three faces are obtained by projecting A on the three coordinate planes. Letting N be the stress normal to the plane BCD, the three components of stress acting parallel to the coordinate axes, are denoted by N_1, N_2, and N_3. The components of force acting in the direction of the coordinates X_1, X_2, X_3 are AN_1, AN_2, and AN_3 respectively. Another useful relationship can be written as:

$$\cos (N_1) = k, \qquad \cos (N_2) = m, \qquad \cos (N_3) = n \tag{1}$$

and the areas of the other faces are Ak, Am, An.

The equations of equilibrium of the tetrahedron can then be written as:

$$\begin{aligned} N_1 &= f_{11}\, k + f_{12}\, m + f_{13}\, n \\ N_2 &= f_{12}\, k + f_{22}\, m + f_{32}\, n \\ N_3 &= f_{13}\, k + f_{23}\, m + f_{33}\, n \end{aligned} \tag{2}$$

The principal stresses for a given set of stress components can be determined by the solution of the following cubic equation:

$$\begin{aligned} f_p^3 - (f_{11} + f_{22} + f_{33})\, f_p^2 + (f_{11} f_{22} + f_{22} f_{33} + f_{11} f_{33} - f_{23}^2 \\ - f_{13}^2 - f_{12}^2)\, f_p - (f_{11} f_{22} f_{33} + 2f_{23} f_{13} f_{12} - f_{11} f_{23}^2 - f_{22} f_{13}^2 - f_{33} f_{12}^2) = 0 \end{aligned} \tag{3}$$

The three roots of this equation give the values of the three principal stresses. The three corresponding sets of direction cosines for the three principal planes can be obtained by substituting each of these stresses (one set for each principal stress) into Equations 3 and using the relation $k^2 + m^2 + n^2 = 1$.

A1.3.2 Specification of Stress at a Point (Cont'd)

$$\begin{aligned} (f_p - f_{11})\, k - f_{12}\, m - f_{13}\, n &= 0 \\ f_{12}\, k + (f_p - f_{22})\, m - f_{23}\, n &= 0 \\ f_{13}\, k - f_{23}\, m + (f_p - f_{33})\, n &= 0 \end{aligned} \tag{4}$$

The shearing stresses associated with the three principal stresses can be obtained by:

$$\begin{aligned} &f^{12} = \pm \tfrac{1}{2}(f_{p1} - f_{p2}),\ f^{13} = \pm \tfrac{1}{2}(f_{p1} - f_{p3}), \\ &f^{23} = \pm \tfrac{1}{2}(f_{p2} - f_{p3}) \end{aligned} \tag{5}$$

where the superscript notation is used to distinguish between the applied shearing stresses and the stresses associated with the principal normal stresses f_{p1}, f_{p2}, and f_{p3}.

The maximum shearing stress acts on the plane bisecting the angle between the largest and the smallest principal stresses and is equal to half the difference between these two principal stresses.

A1.3.3 Equations of Equilibrium

Since no simplifying assumption is permitted as to the distribution of strain in the theory of elasticity, the equilibrium and the continuity of each element within the body must be considered. These considerations are discussed in this and the subsequent subsections.

Let the components of the specific body force be denoted by X_1, X_2, X_3, then the equation of equilibrium in a given direction is obtained by summing all the forces in that direction and proceeding to the limit. The resulting differential equations of equilibrium for three dimensions are:

A1.3.3 Equations of Equilibrium (Cont'd)

$$\frac{\partial f_{11}}{\partial x_1} + \frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + X_1 = 0$$

$$\frac{\partial f_{22}}{\partial x_2} + \frac{\partial f_{12}}{\partial x_1} + \frac{\partial f_{23}}{\partial x_3} + X_2 = 0 \qquad (6)$$

$$\frac{\partial f_{33}}{\partial x_3} + \frac{\partial f_{13}}{\partial x_1} + \frac{\partial f_{23}}{\partial x_2} + X_3 = 0$$

These equations must be satisfied at all points throughout the body. The internal stresses must be in equilibrium with the external forces on the surface of the body. These conditions of equilibrium at the boundary are obtained by considering the stresses acting on Figure A1.3.3-1.

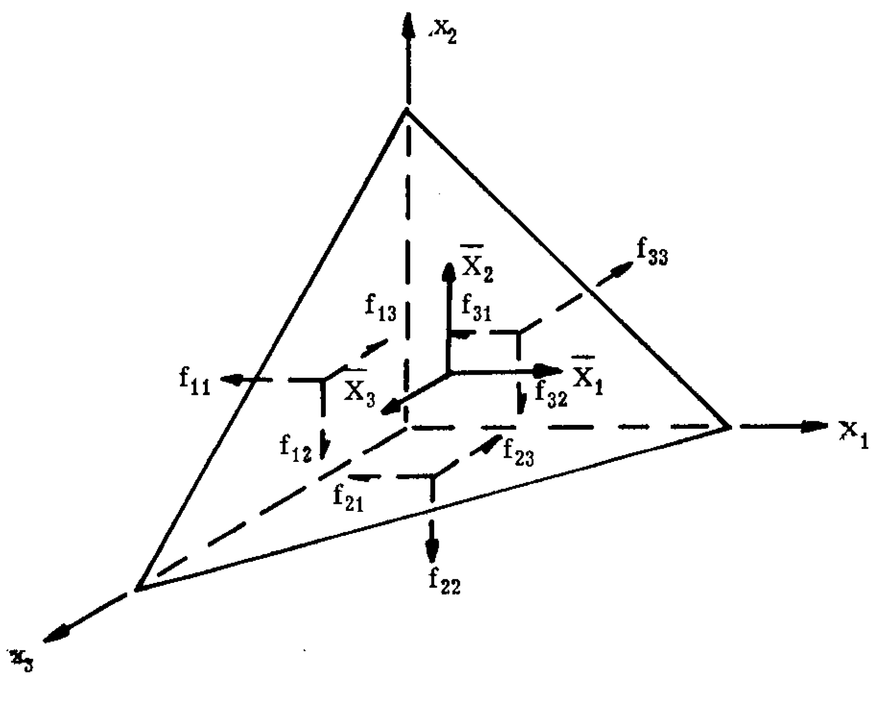

Figure A1.3.3-1 An Element Used in Deriving the Equations of Equilibrium

A1.3.3 Equations of Equilibrium (Cont'd)

By use of Equations 1 and summing forces the boundary equations are:

$$\begin{aligned}
\bar{X}_1 &= f_{11}\,k + f_{12}\,m + f_{13}\,n \\
\bar{X}_2 &= f_{22}\,m + f_{23}\,n + f_{12}\,k \\
\bar{X}_3 &= f_{33}\,n + f_{13}\,k + f_{23}\,m
\end{aligned} \tag{7}$$

in which k, m, n are the direction cosines of the external normal to the surface of the body at the point under consideration and $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ are the components of the surface forces per unit area.

The Equations 6 and 7 in terms of the six components of stress, f_{11}, f_{22}, f_{33}, f_{12}, f_{13}, f_{23} are statically indeterminate. Consideration of the elastic deformations is necessary to complete the description of the stressed body. This is done by considering the elastic deformations of the body.

A1.3.4 Distribution of Strains in a Body

The relations between the components of stress and the components of strain have been established experimentally and are known as Hooke's law. For small deformations where superposition applies, Hooke's law in three dimensions for normal strain is written as:

$$\begin{aligned}
\epsilon_1 &= \frac{1}{E}\,[f_{11} - \nu\,(f_{22} + f_{33})\,] \\
\epsilon_2 &= \frac{1}{E}\,[f_{22} - \nu\,(f_{11} + f_{33})\,] \\
\epsilon_3 &= \frac{1}{E}\,[f_{33} - \nu\,(f_{11} + f_{22})\,]
\end{aligned} \tag{8}$$

A1.3.4 Distribution of Strains in a Body (Cont'd)

and for shearing strain

$$\gamma_{12} = \frac{2(1+\nu)}{E} f_{12} = \frac{f_{12}}{G}$$

$$\gamma_{13} = \frac{2(1+\nu)}{E} f_{13} = \frac{f_{13}}{G} \qquad (9)$$

$$\gamma_{23} = \frac{2(1+\nu)}{E} f_{23} = \frac{f_{23}}{G}$$

These six components of strains can be expressed in terms of the three components of displacements. By considering the deformation of a small element dx_1, dx_2, dx_3 of an elastic body with u, v, w as the components of the displacement of the point 0. The displacement in the x_1 - direction of an adjacent point A on the x_1 axis is

$$u + \frac{\partial u}{\partial x_1} dx_1$$

due to the increase $(\partial u/\partial x_1) dx_1$ of the function u with increase of the coordinate x_1. It follows that the unit elongation at point 0 in the x_1 direction is $\partial u/\partial x_1$. In the same manner it can be shown that the unit elongations in the x_2 - and x_3 - directions are given by $\partial v/\partial x_2$ and $\partial w/\partial x_3$ respectively.

The distortion of the angle from AOB to A'O'B' can be seen from Figure A1.3.4-1 to be $\partial v/\partial x_1 + \partial u/\partial x_2$. This is the shearing strain between the planes $x_1 x_3$ and $x_2 x_3$. The shearing strains between the other two planes are obtained similarly.

The six components of strains in terms of the three displacements are:

$$\epsilon_1 = \frac{\partial u}{\partial x_1}, \qquad \epsilon_2 = \frac{\partial v}{\partial x_2}, \qquad \epsilon_3 = \frac{\partial w}{\partial x_3}$$

$$\gamma_{12} = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}, \qquad \gamma_{13} = \frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1}, \qquad \gamma_{23} = \frac{\partial v}{\partial x_3} + \frac{\partial w}{\partial x_2} \qquad (10)$$

A1.3.4 Distribution of Strains in a Body (Cont'd)

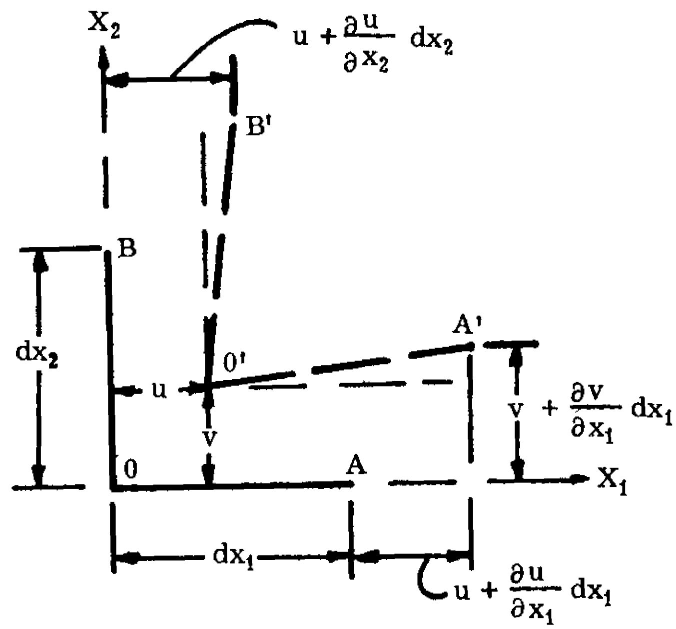

Figure A1.3.4-1 Distortions Due to Normal and Shearing Stresses Used to Define Strains in Terms of Displacements

A1.3.5 Conditions of Compatibility

The conditions of compatibility, that assure continuity of the structure, can be satisfied by obtaining the relationship between the strains in Equations 10. The relationship can be obtained by purely mathematical manipulation as follows:

Differentiating ϵ_1 twice with respect to x_2; ϵ_2 twice with respect to x_1; and γ_{12} once with respect to x_1 and once with respect x_2. The sum of the derivatives of ϵ_1 and ϵ_2 is found to be identical to the derivative of γ_{12}. Therefore,

$$\frac{\partial^2 \epsilon_1}{\partial x_2^2} + \frac{\partial^2 \epsilon_2}{\partial x_1^2} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2}$$

A1.3.5 Conditions of Compatibility (Cont'd)

Two more relationships of the same kind can be obtained by cyclic interchange of the subscripts 1, 2, 3.

Another set of equations can be found by further mathematical manipulation as follows:

Differentiate ϵ_1 once with respect to x_1 and once with respect to x_3; γ_{12} once with respect to x_1 and once with respect to x_3; γ_{13} once with respect to x_1 and once with respect to x_2; and γ_{23} twice with respect to x_1. It then follows that

$$2\,\frac{\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_3} + \frac{\partial^2 \gamma_{13}}{\partial x_1 \partial x_2} - \frac{\partial^2 \gamma_{12}}{\partial x_1^2}$$

Two additional relationships can be found by the cyclic interchange of subscripts as before.

The six differential relations between the components of strain are called the equations of compatibility and are given below.

$$\frac{\partial^2 \epsilon_1}{\partial x_2^2} + \frac{\partial^2 \epsilon_2}{\partial x_1^2} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2}, \quad \frac{2\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial \gamma_{12}}{\partial x_3} + \frac{\partial \gamma_{13}}{\partial x_2} - \frac{\partial \gamma_{22}}{\partial x_1}\right),$$

$$\frac{\partial^2 \epsilon_2}{\partial x_3^2} + \frac{\partial^2 \epsilon_3}{\partial x_2^2} = \frac{\partial^2 \gamma_{33}}{\partial x_2 \partial x_3}, \quad \frac{2\partial^2 \epsilon_2}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2}\left(\frac{\partial \gamma_{12}}{\partial x_3} - \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{23}}{\partial x_1}\right), \qquad (11)$$

$$\frac{\partial^2 \epsilon_3}{\partial x_1^2} + \frac{\partial^2 \epsilon_1}{\partial x_2^2} = \frac{\partial^2 \gamma_{13}}{\partial x_1 \partial x_3}, \quad \frac{2\partial^2 \epsilon_3}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(-\frac{\partial \gamma_{12}}{\partial x_3} + \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{23}}{\partial x_1}\right)$$

These equations of compatibility may be stated in terms of the stresses if the strains in Equations 11 are expressed in terms of the stresses by Hooke's law (Equations 8 and 9). Differentiating each of Equations 8 and 9 as required for substitution, we have

A1.3.5 Conditions of Compatibility (Cont'd)

$$(1+\nu)\ \nabla^2 f_{11} + \frac{\partial^2\theta}{\partial x_1^2} = 0 \quad , \qquad (1+\nu)\ \nabla^2 f_{23} + \frac{\partial^2\theta}{\partial x_2 \partial x_3} = 0$$

$$(1+\nu)\ \nabla^2 f_{22} + \frac{\partial^2\theta}{\partial x_2^2} = 0 \quad , \qquad (1+\nu)\ \nabla^2 f_{13} + \frac{\partial^2\theta}{\partial x_1 \partial x_3} = 0 \qquad (12)$$

$$(1+\nu)\ \nabla^2 f_{33} + \frac{\partial^2\theta}{\partial x_3^2} = 0 \quad , \qquad (1+\nu)\ \nabla^2 f_{12} + \frac{\partial^2\theta}{\partial x_1 \partial x_2} = 0$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

and

$$\theta = f_{11} + f_{22} + f_{33}$$

For most cases where strains are linear and superposition applies, the system of Equations 6, 7, and 11 or 12 are sufficient to determine the stress components without ambiguity. The use of stress functions to aid in the solution of these equations are discussed below.

A1.3.6 Stress Functions

It has been shown in the previous sections that the differential equations of equilibrium (Equations 6) ensure a distribution of stress in a body that preserves the equilibrium of every element in the body. The fact that these are satisfied does not necessarily mean that the distribution of stresses are correct since the boundary stresses must also be satisfied. The compatibility equations (Equations 11) must also be satisfied to ensure the proper strain distribution throughout the body. The problem is then to find an expression that satisfies all

A1.3.6 Stress Functions (Cont'd)

these conditions. The usual procedure is to introduce a function called a stress function that meets this requirement. For the sake of simplicity, this section will deal only with problems in two dimensions. The stresses due to the weight of the body will also be neglected.

In 1862, G. B. Airy introduced a stress function ($\phi(x_1, x_2)$) which is an expression that satisfies both Equations 6 and 11 (in two dimension) when the stresses are described by:

$$f_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \quad , \quad f_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \quad , \quad f_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \tag{13}$$

By operating on Equations 13 and substituting into Equations 11, we find that the stress function ϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = \nabla^4 \phi = 0 \tag{14}$$

Thus the solution of a two-dimensional problem reduces to finding a solution of the biharmonic equation (Equation 14) which satisfies the boundary conditions (7) of the problem.

A1.3.7 Use of Equations from the Theory of Elasticity

Proficiency in the use of stress functions is gained mainly by experience. It is not unusual to find an expression that satisfies Equation 14 first and then try to determine what problem it solves.

The following problem is presented to illustrate the basic procedure in the use of stress functions.

A1.3.7 Use of Equations from the Theory of Elasticity (Cont'd)

Statement of the problem:

Determine the stress function that corresponds to the boundary conditions for a cantilever beam of rectangular cross section of unit width and loaded as shown in Figure A1.3.7-1. From this stress function determine the stresses and cómpare with the maximum flexure stresses as obtained by the method of mechanics.

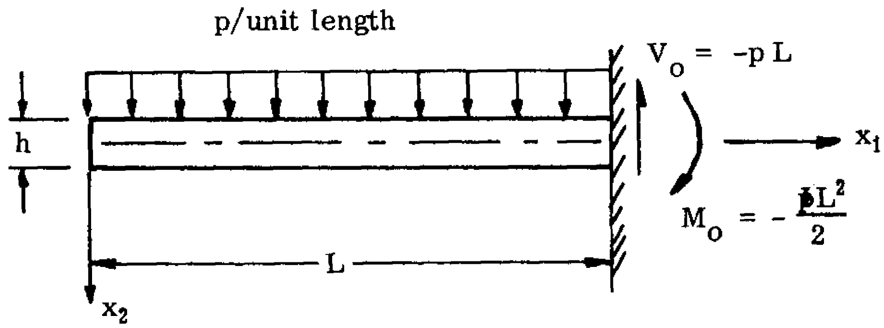

Figure A1.3.7-1 Sample Problem

Solution:

Assume that the stress function is

$$\phi = ax_2^5 + bx_2^3x_1^2 + cx_2^3 + dx_2x_1^2 + ex_1^2$$

Operate on ϕ to satisfy Equation 14

$$\nabla^4\phi = (5\cdot 4\cdot 3\cdot 2)\, ax_2 + 2(3\cdot 2\cdot 2\, bx_2) = 0$$

$$24\, x_2\, (5a + b) = 0$$

from which $\underline{\underline{a = -b/5}}$ (a)

A1.3.7 Use of Equations from the Theory of Elasticity (Cont'd)

Since Equation 14 can now be satisfied by letting $a = -b/5$, the only other condition to satisfy is the boundary conditions.

From Figure A1.3.7-1 the boundary conditions are as follows:

1. $f_{22} = -p \quad \text{at} \quad x_2 = -h/2$

2. $f_{22} = 0 \quad \text{at} \quad x_2 = h/2$

3. $\int_{-h/2}^{h/2} f_{12}\, dx_2 = -pL \quad \text{at} \quad x_1 = L \quad \text{from} \quad \Sigma F = 0$

4. $\int_{-h/2}^{h/2} f_{11} x_2\, dx_2 = -p L^2/2 \quad \text{at} \quad x_1 = L \quad \text{from} \quad \Sigma M = 0$

5. $f_{12} = 0 \quad \text{at} \quad x_2 = h/2$

From Equation 13

$$f_{11} = \frac{\partial^2 \phi}{\partial x^2} = 20ax_2^3 + 6bx_1^2 x_2 + 6cx_2$$

$$f_{22} = \frac{\partial^2 \phi}{\partial x_1^2} = 2bx_2^3 + 2hx_2 + 2e \qquad \text{(b)}$$

$$f_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -6bx_2^2 x_1 - 2hx_1$$

Using boundary condition 1

$$f_{22} = -p = -\frac{2bh^3}{8} - \frac{2dh}{2} + 2e \qquad \text{(c)}$$

A1.3.7 Use of Equations from the Theory of Elasticity (Cont'd)

from boundary condition 2

$$f_{22} = 0 = \frac{2bh^3}{8} + \frac{2dh}{2} + 2e \qquad \text{(d)}$$

adding (c) and (d)

$$4e = -p \quad \text{or} \quad \underline{\underline{e = -p/4}} \qquad \text{(e)}$$

from boundary condition 3

$$\int_{-h/2}^{h/2} f_{12}\,dx_2 = \int_{-h/2}^{h/2} [-6bx_2^2x_1 - 2hx_1]\,dx_2$$

$$= 2\left[-\frac{6}{3}\,bLx_2^3 - 2hLx_2\right]_0^{h/2} = -pL \qquad \text{(f)}$$

or

$$\underline{\underline{\frac{bh^3}{2} + 2dh \doteq p}}$$

from boundary condition 4

$$\int_{-h/2}^{h/2} f_{11}\,x_2\,dx_2 = \int_{-h/2}^{h/2} [20ax_2^4 + 6bx_1^2x_2^2 + 6cx_2^2]\,dx_2$$

$$= 2\left[\frac{20a}{5}\,x_2^5 + \frac{6b}{3}\,x_1^2x_2^3 + \frac{6}{3}\,cx_2^3\right]_0^{h/2}$$

$$= \frac{ah^5}{4} + \frac{bL^2h^3}{2} + \frac{ch^3}{2} = -pL^2/2$$

A1.3.7 Use of Equations from the Theory of Elasticity (Cont'd)

substituting Equation a and solving for c

$$c = \frac{-pL^2 - b\,(L^2h^3 - h^5/10)}{h^3} \qquad (g)$$

from boundary condition 5

$$f_{12} = \frac{-6}{4}\,bh^2x_1 - 2dx_1 = 0$$

$$= -x_1\left(\frac{3}{2}\,bh^2 + 2d\right)$$

or

$$b = \frac{-4d}{3h^2} \qquad (h)$$

Solving Equations f and h simultaneously we get

$$d = \frac{3p}{4h} \quad \text{and} \quad b = -p/h^3 \qquad (i)$$

Substituting $b = -p/h^3$ into Equation g

$$c = -\frac{p}{10L} \qquad (j)$$

A1.3.7 Use of Equations from the Theory of Elasticity (Cont'd)

The stress function can now be written as

$$\phi = -px_1^2 (x_2^3/h^3 - 3x_2/4h + 1/4) + (ph^2/5)(x_2^5/h^5 - x_2^3/2h^3) \quad (k)$$

and the stresses as (see Equations b)

$$f_{11} = -\frac{p}{2I}(x_1^2 x_2 + h^2 x_2/10 - 2x_2^3/3) \quad (l)$$

$$f_{22} = -\frac{p}{2I}(x_2^3/3 - h^2 x_2/4 + h^3/12) \quad (m)$$

$$f_{12} = \frac{p}{2I}(x_2^2 x_1 - h^2 x_1/4) \quad (n)$$

where $I = h^3/12$

Comparison of maximum flexure stresses from Equation l with $x_1 = L$, $x_2 = -h/2$

$$f_{11}^{\text{elasticity}} = \frac{ph}{4I}\left(L^2 - \frac{h^2}{15}\right) \quad (o)$$

from elementary mechanics

$$f_{11}^{\text{mechanics}} = \frac{Mc}{I} = \frac{pL^2}{4I}h \quad (p)$$

The difference is then

$$f_{11}^{\text{elasticity}} - f_{11}^{\text{mechanics}} = -\frac{ph^3}{60I} = \underline{\underline{-\frac{p}{5}}} \quad (q)$$

A1.4.0 Theories of Failure

Several theories have been advanced to aid in the prediction of the critical load combination on a structural member. Each theory is based on the assumption that a specific combination of stresses or strains constitutes the limiting condition. The margin of safety of a member is then predicted by comparing the stress, the strain, or combination of stress and strain with the corresponding factors as determined from tests on the material.

Three of the more useful theories are stated in this subsection. A more detailed discussion on these and other theories of failure can be found in most elementary strength analysis text books such as references 2 and 3.

The Maximum Normal Stress Theory

The maximum normal stress theory of failure states that inelastic action at any point in a material begins only when the maximum principal stress at the point reaches a value equal to the tensile (or compressive) yield strength of the material as found in a simple tension (or compression) test. The normal or shearing stresses that occur on other planes through the point are neglected.

The Maximum Shearing Stress Theory

The maximum shearing stress theory is based on the assumption that yielding begins when the maximum shear stress in the material becomes equal to the maximum shear stress at the yield point in a simple tension specimen. To apply it, the principal stresses are first determined, then, according to Equation 5,

$$f_{max}^{ij} = \frac{1}{2}(f_{pi} - f_{pj})$$

where i and j are associated with the maximum and minimum principal stresses respectively.

The Maximum Energy of Distortion Theory

The maximum energy of distortion theory states that inelastic action at any point in a body under any combination of stresses begins only when the strain energy of distortion per unit volume absorbed at the point is equal to the strain

A1.4.0 Theories of Failure (Cont'd)

energy of distortion absorbed per unit volume at any point in a bar stressed to the elastic limit under a state of uniaxial stress as occurs in a simple tension (or compression) test. The value of this maximum strain energy of distortion as determined from the uniaxial test is

$$w_1 = \frac{1+\nu}{3E} F_{yp}^2$$

and the strain energy of distortion in the general case is

$$w = \frac{1+\nu}{6E} [(f_{p1} - f_{p2})^2 + (f_{p2} - f_{p3})^2 + (f_{p1} - f_{p3})^2]$$

where f_{p1}, f_{p2}, f_{p3} are the principal stresses and F_{yp} is the yield point stress. (For the case of a biaxial state of stress, $f_{p3} = 0$.)

The condition for yielding is then, $w = w_1$ or

$$(f_{p1} - f_{p2})^2 + (f_{p2} - f_{p3})^2 + (f_{p1} - f_{p3})^2 = 2 F_{yp}^2$$

A1.4.1 Elastic Failure

The choice of the proper theory of failure is dependent on the behavior of the material. It is suggested that the maximum principal stress theory be used for brittle materials and either the maximum energy of distortion theory or the maximum-shearing-stress theory for ductile materials.

The choice between the two methods for ductile materials may be made by considering the particular application. When failure of the component leads to catastrophic results, the maximum-shearing-stress theory should be used since the results are on the safe side.

A1.4.2 Interaction Curves

No general theory exists which applies in all cases for combined loading conditions in which failure is caused by instability. Interaction curves for the instability case or other critical load conditions are usually determined from or substantiated by structural tests. The analysis of various loading combinations are discussed in Section A3.

A1.0.0 Stress and Strain

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